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Aggregation of experts' opinions and conditional consensus opinion

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Abstract

This paper considers aggregation of experts opinions expressed through not necessarily independent but even conflicting probability distributions or multiple priors. The paper addresses elicitation of common opinion and derives its conditional probabilities on future events.

1 Introduction

Combining experts opinions represented by probability distributions is a multidisciplinary process, that involves mathematical and behavioral approaches, for eliciting a consensus distribution or common probability distribution. Under uncertainty or ambiguity the decision-maker (DM henceforth) has to take an action on the basis of aggregated probabilities by considering not only opinions of who he regards as experts but also competence or quality of their judgements. Savage (1971) set that "risks characterized by tiny probabilities may be divergent, and, what is more relevant, you might discover which expert is optimistic or pessimistic in some respect and therefore temper his judgements. Should he suspect you of this, however, you and he may be on the escalator of perdition"[1].

In the Savage's perspective, mathematical Bayesian aggregation models manage individual probability distributions to obtain a single combined one operating with different degree of complexity to represent experts' different competence, reliability and independence, such that: equal weight, best expert, copula etc. (Stone 1961; Genest and Zidek 1986; Cooke 1991; Ramanathang and Genesh 1994; Plous 1993). Over the last decade there has been large attention on determining weights of experts in the group decision making process, in the Bayesian axiomatic approach to consensus distribution: i.e. Yue 2012; Xu and Kai 2012; Abootalevi et al 2018.

Under uncertainty, the Bayesian axiomatic approach to consensus distribution does not admit a consistent representation of experts' incomplete, fuzzy,

[1] In 1954, in the lecture at the Istituto Universitario di Studi Europei in Turin, de Finetti considers a decision making process involving experts. Given the dilemma of choice between "mean of decisions or mean of opinions", de Finetti sets the latter, even if it conveys the choice of a particular type of mean when experts have different competence or expertise.

not reliable and conflicting information. Some methods, different from Bayesian pool operators, were proposed to readjust imprecise or ambiguous opinions of experts based on closed and convex set of probabilities or capacities: Cres et al 2011; Gajdos and Vergnaud 2013; Basili and Pratelli 2015.

This paper presents an approach to elicit consensus distribution among not necessarily independent and fully competent experts based on multiple priors. The paper focuses on the Steiner point and by considering its properties, with a class of probability measures, directly derives conditional judgements, that are particular relevant when the DM has to evaluate consequences related to rare events.

The paper proceeds as follows. Section 2 gives main motivation and the related literature. Section 3 introduces notation and definition. The Steiner point is defined and the consensus distribution is elicited. An example makes clear the Steiner point evaluation. Section 4 solves the updating problem of the consensus opinion when learning is considered. Section 5 concludes.

2 Main motivation

A fundamental assumption of Bayesianism is that a rational agent has complete certainty about the probabilities of states of the World that is represented by a unique, additive and fully reliable probability distribution. In strict Bayesianism, subjective probabilities are derived by bets about states (Ramsey 1926, de Finetti 1936, Savage 1954) and the Dutch Book theorem, that does not allow arbitrage, implies that agent is willing to take either sides of a bet: if the agent accepts a bet on state i at odds of $c : d$, she accepts a bet on $\neg i$ ($\neg$ is the complement operator) at odds $d : c$. Nevertheless, agents facing uncertainty could contradict coherence criterion and rejects one side of the bet. In fact, the Dutch Book theorem assumes a Boolean algebra of events, but DM can be unable to have a fully reliable probability distribution on events making Bayesian conditional inference incoherent.

Given possible future rare events, that are often disregarded or considered 'outlier', a policy-maker could be interested in assessing their likelihood; however experts could be not able to elicit a reliable probability distribution but, at most, an interval of probabilities or even an ordinal judgment such as: low probable, high probable etc.

Nevertheless, there exist events that are very rare but could induce catastrophic risk for human beings. Pandemic flu, climate change induced by global warming and asteroid crash into Earth are examples of potentially catastrophic events and may happen that the DM have to decide how to manage them. In Bayesian theory, this problem is solved by conditional probability and the DM could choose among alternatives by applying maximum expected utility principle that only requires additive decomposable utility function.

Sometime data or evidence are inadequate or unreliable to elicit probabilities interpreted as betting rates in risk neutral framework. Experts could be not able to elicit a reliable probabiliy that represent the tipping points (abrupt state changes or large-scale singular events) in climate or ecosystems. As an example, they could be not able to assess a reliable response of the climate system to possible trajectories of radiative forcing from aerosols such as: Atlantic Meridional Overturning Circulation – AMOC, the Greenland Ice Sheet – GIS, the West Antarctic Ice Sheet - WAIS, the Amazon Rain Forest and the El Nino/Southern Oscillation – ENSO (Kriegler et al. 2009).

To overcome this unsatisfactory situation, the DM could use expert elicitation protocols, such as Delphi, Q-Methodology, Nominal Group technique, Kaplan approach etc, but all of them suffer from many problems: polarization, strategic manipulation, overconfidence, self-censorship, pressure to conform, anchoring, adjustment, etc. (US EPA 2011). Moreover, experts' ambiguity could be very large or express ignorance, even if assessment is represented by interval of probabilities (Zickfeld et al. 2007) or qualitative scale (IPCC 5th Assessment Report 2014)[2].

Nonetheless, in the case of radiative forcing and climate response resulting from atmospheric aerosol concentration that scatter or absorb solar radiation and modify cloud properties by altering the radiation budget, the DM could be interested to evaluate climate response to alternative future trajectories of radiative forcing[3].

3 Notation and definition

Let S be the finite set of states of the world, where $S = \{s_1, s_2, ..., s_n\}$, Σ be the *algebra* of events where $\Sigma = 2^S$ and P be a probability distributions on (S, Σ). A measure $\upsilon \geq 0$ is a positive capacity on (S, Σ) if $\upsilon : A \in \Sigma \to \upsilon(A) \in \mathbb{R}$, where $\upsilon(\varnothing) = 0$, $\upsilon(S) = 1$ and $A, B \in \Sigma$ such that $A \subseteq B \Longrightarrow \upsilon(A) \leq \upsilon(B)$. A capacity υ is convex if $\upsilon(A \cup B) + \upsilon(A \cap B) \geq \upsilon(A) + \upsilon(B)$. The dual capacity[4] $\overline{\upsilon}$ of a capacity υ is defined by $\overline{\upsilon}(A) = 1 - \upsilon(A^C)$ $\forall A \in \Sigma$. Uncertainty is modeled through the *core* of a convex capacity υ, i.e., through the set $C(\upsilon)$ of

[2] For decisions under risk and uncertainty, the core team of scientists set a guide note intended to assist Lead Authors of the 5th Assessment Report of IPCC on Climate Change. The guidance note on consistent treatment of uncertainties "rely on two metrics for communicating the degree of uncertainty in key findings: confidence in the validity of finding based on the type, amount, quality and consistency of evidence (e.g. mechanistic understanding, theory, data, models, expert judgment) and the degree of agreement" (IPCC working-group for the 5th Assessment Report AR5). The team defines ordinal scales to represent the validity of findings (limited, medium, robust), the level of confidence, that has not be interpreted probabilistically, of the author teams' judgements about the validity of findings (very low, low, medium, high, very high) and likelihood or a calibrated language for describing quantified uncertainty (exceptionally unlikely, very unlikely, unlikely, about as likely as not, likely, very likely, virtually certain).

[3] Unfortunately, the behavior of climate system is very uncertain and ambiguous probabilistic estimations of equilibrium climate sensitivity result from models of different complexity and statistical methods. Expert judgments about global mean temperature response to different forcing trajectories show a considerable variation in the increase of average temperature in three corridors from1,5C^o to 6C^o (Kriegler et al. 2009), and "the ordering of mean ranks is not entirely robust with respect to the procedure used" (Zickfeld et al. 2010).

[4] If the capacity υ is convex its dual capacity $\overline{\upsilon}$ is concave.

probability distributions P on (S,Σ) above υ, or $P(A) \geq \upsilon(A)\ \forall A \in \Sigma$.

For $X : \Sigma \rightarrow \mathbb{R}$, an act, the cumulative distribution function F_X of X with respect to a probability P is defined by $x \in \mathbb{R} \rightarrow F_X(x) = P(X \leq x)$. For every X and υ, the Choquet integral of X with respect to υ denoted $\int X d\upsilon$ is defined by $\int X d\upsilon = \int_{-\infty}^{0} (\upsilon(X \geq t) - 1)dt + \int_{0}^{+\infty} \upsilon(X \geq t)dt$.

3.1 The Steiner Point

The DM asks to a finite set of experts $j = 1, ..., m$ to value the possible probability distributions P governing an uncertain phenomenon that may occur in the future. It is assumed that there is a unique unknown probability distribution P_0 governing the phenomenon. Each expert i, will be asked to give *lower* and *upper bounds* for the probability $p_0^i = P_0(\{i\})$, or probability envelope, so that the resuling probability interval is a convex set with a finite set of extreme points (Kyburg 1987). The set of possible probabilities P_j considered by expert j will be $P_j = \left\{ P = (p_1, ..., p_i, ..., p_m),\ a_i^j \leq p_i \leq b_i^j,\ i = 1, ..., n \right\}$ and $0 \leq a_i^j \leq b_i^j \leq 1$. It is straightforward that $P_j \neq \varnothing$, i.e. proper, if and only if (de Campos et al. 1994):

$$\sum_i a_i^j \leq 1 \leq \sum_i b_i^j \qquad [1].$$

If the set of all possible probability distributions of expert j is not empty ($P_i \neq \varnothing$), P_j can be regarded as the the *core* $C(\upsilon_j)$ of a convex capacity υ_j such that $\upsilon_j(A) = Max\left(\sum_{i \in A} a_i^j,\ 1 - \sum_{i \notin A} b_i^j \right)$ (Chateauneuf and Cornet 2012). Even if experts do not know P_0 they should contemplate a set $\mathcal{P}_j$, such that $P_0 \in \mathcal{P}_j$. By competence and reliability of the experts it is reasonable to expect that $\bigcap_j \mathcal{P}_j \neq \varnothing$, that is the experts should not have fully conflicting opinions. From [1], it is immediate that $\bigcap_j \mathcal{P}_j \neq \varnothing$ is equivalent to $\sum_i a_i \leq 1 \leq \sum_i b_i$ and $a_i \leq b_i\ \forall i$, where: $a_i = \underset{j}{Max}\, a_i^j$, $b_i = \underset{j}{\min}\, b_i^j$ (Basili and Chateauneuf 2016).

On the contrary, if the experts have imprecise and fully conflicting opinions, i.e. $\bigcap_j \mathcal{P}_j = \varnothing$, the DM should require them to revise their opinion by reconsidering their $\mathcal{P}_j$, in order to satisfy the consistency requirement $\bigcap_j \mathcal{P}_j \neq \varnothing$. If the minimal consistency condition holds, the *consensus opinions* $P \bigcap_j \mathcal{P}_j$ can be defined through the convex capacity υ such that $\upsilon(A) = Max\left(\sum_{i \in A} a_i,\ 1 - \sum_{i \notin A} b_i \right)$ and the convex capacity υ can now be considered as the aggregation of the multiple prior opinions.

In such a framework the Steiner point has a particular interest. The Steiner point or curvature centroid of a convex set is the centroid of a system of masses attached to its vertices[5]. The Steiner point is additive, uniformly continuous and satisfies invariance property respect to isometries (Shephard 1966; Berg 1971). The Steiner point $\Pi^{St} \in C(v)$ can be considered as the representative probability of the consensus experts' opinions. As a matter of fact the Steiner point is defined as the *center of* $P \in C(v)$, so as a meaningful probability summarizing the consensus experts' opinions.

It is well known that the core of a convex capacity can be represented as the core of a convex cooperative game with transferable utility (TU). The core of a convex TU is non empty and the Shapley value of such a balanced, i.e. with non-empty core, game belongs to it (Shapley 1971). Moreover the extreme points of the core of a convex game are its marginal worth vectors; so the core coincides with the Weber set or the closed convex set of marginal vectors[6]. In such a situation, the Shapley value coincides with the Steiner point of its core[7] (Gajdos et al 2008, Pechersky 2015), moreover the Shapley value of an arbitrary convex TU game can be represented as a difference of the Steiner points of the cores of two convex game (Rosenmuller 1981, Pechersky 2012). Computation

of the Shapley value (Owen 1995, 265) is easier than evaluation of the Steiner point and the Shapley value Π^{Sh} (i.e. $\Pi^{Sh} = \Pi^{St}$) can be determined as follows:

$$\forall i \in [1,n] \;\; \Pi_i^{St} = \sum_{i \in A \subset S} \tfrac{(|A|-1)!(n-|A|)!}{n!} \left[v(A) - v(A \backslash \{i\}) \right] \qquad [2].$$

Example 1 *Computation of the Shapley value for the following 'probability-interval' capacity* v

$$\begin{array}{lcccc} S & = & s_1 & s_2 & s_3 \\ b_i & = & \frac{6}{14} & \frac{5}{14} & \frac{7}{14} \\ a_i & = & \frac{2}{14} & \frac{3}{14} & \frac{4}{14} \end{array}$$

therefore v is given by

[5] The Steiner point of a core is the weighted average of its vertices, in which the weight for each vertex is proportional to its outer angle.

Formally, given a convex compat set $A \in R^n$, the Steiner point $St(A) = \frac{1}{\sigma_n} \int\limits_{S^{n-1}} up(u, A) dw$

where u is a variable unit vector, $p(u, A)$ is the value of support function of A in direction u, dw is an element of surface area of the unit sphere S^{n-1}, and σ_n is the content of the $n - dimensional$ unit ball (Pechersky 2015).

[6] Webber (1978) proved that for any game the core is included into the convex hull of its marginal worth vectors and for a convex game they coincide (Shapley 1971).

[7] "An external angle of a polytope A at a vertex z is defined as the ratio of the (n-1)-content of the intersection of the normal cone to A at z with the unit (n-1)-sphere S^{n-1}centered on the origin to the (n-1)-content of S^{n-1}"(Pechersky 2015, 490). Here, the vertices of the core are precisely the marginal worth vectors and the external angle at a vertex of the core is proportional to the number of marginal worth vectors defining this vertex. Then "the Shapley value of a convex game is a weighted sum of the extreme points of the core with the weights equal to $\frac{k(x)}{n!}$" (Pechersky 2015, 492); where N is the player set, $k \in N$, $k = \{1, 2, ..., n\}$ and $k(x)$ is the number of marginal vectors defining the extreme point x.

A	$\{s_1\}$	$\{s_2\}$	$\{s_3\}$	$\{s_1, s_2\}$	$\{s_1 s_3\}$	$\{s_2 s_3\}$	S
$v(A)$	$\frac{2}{14}$	$\frac{3}{14}$	$\frac{4}{14}$	$\frac{5}{14}$	$\frac{7}{14}$	$\frac{7}{14}$	$\frac{14}{14}$

then by [2]:

$\Pi_1^{\vartheta} = \frac{1}{6}\left\{2.\frac{2}{14} + \frac{1.(5-3+7-4)}{14} + \frac{2.(14-7)}{14}\right\} = \frac{23}{84}$

$\Pi_2^{\vartheta} = \frac{1}{6}\left\{2.\frac{3}{14} + \frac{1.(5-2+7-3)}{14} + \frac{2.(14-5)}{14}\right\} = \frac{31}{84}$

$\Pi_3^{\vartheta} = \frac{1}{6}\left\{2.\frac{4}{14} + \frac{1.(7-2+7-4)}{14} + \frac{2.(14-7)}{14}\right\} = \frac{30}{84}$

hence $\Pi_i^{\vartheta} = (\frac{23}{84}, \frac{31}{84}, \frac{30}{84})$.

4 Conditional consensus opinion

Assume that the process that makes information available to the DM is represented by the occurrence of a fixed non-null event Γ. What is the conditional consensus opinion elicited when Γ occurs? The standard solution is of course the Bayesian updating, but unfortunately, it is not possible to update the Steiner point by the Bayes rule, because of dynamic incosistency[8].

Literature about multiple priors updating sets some rules, the simplest of which is to apply the Bayes rule to each probability distributions in the core (prior-by-prior updating) and reassess the Steiner point by applying [2]. Bayesian updating of the core ensures dynamic consistency, but the re-evaluation of the Steiner point can be laborious, and some alternative method are possible even if assumptions have to be introduced: rectangularity[9], menu dependence etc.

However Jaffray (1992) and Chateauneuf et al. (2011) define an updating process which satisfies some desirable properties. Since a convex capacity is the lower envelope of its core, then if v^{Γ} represents $(C(v))^{\Gamma}$ for any non-null event Γ, the updating is regular. Jaffray (1992) shows that if S is a finite set, a capacity v satisfies regular updating if and only if:

$$\left[\begin{array}{c} A, B \in \Sigma \setminus \{\varnothing, S\} \\ 0 < v(A \cap B), v(A \cup B) < 1 \end{array}\right] \Longrightarrow [v(A \cap B) + v(A \cup B) = v(A) + v(B)] \qquad [3].$$

This condition is not always satisfied, but there is a set of convex, regular and strictly positive (i.e. $v > 0$) capacities, indeed a parametric class of capacities introduced by Huber (1981) called $(\epsilon, \delta)-$*contamination* , that verifies required conditions. Among $(\epsilon, \delta) -$ *contamination* capacities there is a special class indeed the $\epsilon-$*contamination* capacities $(\delta = 0)$[10] that not only satisfies the full

[8] Dynamic consistency requires that updated preferences are consistent with ex-ante preference, i.e. ex-post preference is the same as the DM's ex-ante preference.

[9] Since dynamic consistency implies the decomposition of a probability distribution in term of its conditionals and marginals, "a set of prior is rectangular if its induced sets of conditionals and marginals admit a corresponding decomposition" (Epstein and Schneider 2003).

[10] The $\epsilon-$contamination structure in the empirical Bayes analysis of individual beliefs or robust Bayesian analysis was introduced by Huber (1964, 1965). The Robust Bayesian viewpoint affirms that one of the main justification for using Bayesian analysis is that prior distributions can never be quantified or elicited exactly (i.e., without error), especially in a finite amount of time. A probability distribution can be contaminated by $\epsilon \in [0, 1]$ another one. The parameter ϵ reflects the amount of error that is considered possible.

bayesian updating rule, but also the Dempster-Shafer updating rule, that is the most famous updating rule for beliefs and possibility measures[11]. Crucially an $\epsilon - contamination$ probability distribution is a convex capacity, i.e. $\forall A \subseteq \Sigma$, $v_j(A) = \epsilon \bar{P}_j(A)$ if $A \neq \Sigma$. In this perspective, the consensus distribution elicited by the Steiner point $\Pi^{St} \in C(v)$ is nothing less than a proxy of the true not know P_0 and the parameter $\epsilon \in [0, 1)$ can be considered as the error in experts' approximation or $\Pi^{St} = \epsilon P_0$, and it is possible to apply the full Bayesian updating rule. Moreover (Chateauneuf et al. 2011), if S is an infinite countable set and v is strictly positive, weakly lower continuous and convex, then v satisfies regular updating property and v is an $\epsilon - contamination$ with $\varepsilon \in [0, 1)$.

Then $(\Pi^{St})^{\Gamma} = \Pi^{St}_{\Gamma}$ [4].

By [4] it is possible to update the Steiner point only, instead of all probabilities belonging to the core, and eliciting the consensus opinion of experts conditional to any event in Σ.

Regular updating gives the possibility to evaluate the conditional consensus opinion, defined as the Steiner point, directly and easily. This is a usefull possibility when DM has to consider uncertain climate change consequences, that in the case of average increase temeratures implies evaluation of risk and potential for adaptation with respect to global events (i.e. crops yields and increases in yield variability, increase morbidity and mortality, reduced access to water, etc.) induced by climate-related drivers of impacts such as: warming trend, extreme temperature, drying trend, extreme precipitation, damaging cyclone, flooding, storm surge, ocean acidification, carbon dioxide fertilization, reduction in terrestrial carbon sink, Boreal tipping point, Amazon tipping point, species extinction, marine biodiversity loss etc.

5 Concluding remarks

The paper studies the problem of aggregating probabilities for a given set of possible, sometimes rare, future events by uncertain, not necessarily independent but even conflicting experts. The paper offers a simple method to calculate a single probability distribution taking into account different competence and credibility of experts. Uncertain experts' options are represented by closed and convex set of probability distributions (multiple priors) or capacities, none of which is considered fully reliable. The paper introduces the Steiner point as a representation of experts' consensus opinion and shows how the Steiner point can be updated by learning. If the experts' opinions are $\epsilon - contamination$

[11] Given $A, B \in \Sigma$, Dempster-Shafer updating rule is $v(A \mid B) = \frac{v(A \cup B^C) - v(B^C)}{1 - v(B^C)}$ where $v(B^C) = 1 - v(B)$.

If S is a finite set and v is a strictly convex capacity on Σ, "the following statements are equivalent:

(i) v satisfies the Dempster and Shafer Consistency Property for full bayesian updating rule;

(ii) v is an $\varepsilon - contamination$ with $\varepsilon \in [0, 1)$" (Chateauneuf et al 2011, 120).

capacities the conditional Steiner point is elicited by the simple full bayesian updating rule.

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