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Sufficientarian Grading Rules and Rankings: Characterizations and Implementation

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Abstract

Sufficientarian grading rules are defined using a finite family of sufficientarian judgements on individual capability assignments as embodied in a sufficientarian binary grading function (BGF). Both sufficientarian grading rules and the sufficientarian total preorders on capability-type assignments they induce are characterized. Moreover, several further total preorders based upon sufficiency-gap information provided by a sufficientarian grading rule are explicitly defined and some of them are also characterized. It is also shown that there exists a class of inclusive, unanimity-respecting and suitably strategy-proof protocols (including simple majority when the number of agents is odd) which can be deployed in order to select one specific sufficientarian grading rule.

Keywords: *Sufficientarianism, Grading Function, Thresholds, Rating, Ranking*

JEL Classification: D31, D63.

1 Introduction

In the last few decades a considerable amount of work has been devoted to *sufficientarianism*, the class of distribution rules establishing that '*everyone should have enough*' to live a properly accomplished and socially respected life, or 'enjoy sufficient freedom' as someone might perhaps like to express that very notion. Thus, any such sufficientarian rule also provides in a most straightforward way both *ratings* as based on *benchmarks* (e.g. sets of goals or targets) and the resulting *rankings* (e.g. preorders, including of course *total* preorders). Such ratings and rankings are meant to enable, respectively, '*intrinsic*' assessments of *individual* assignments and of the resulting *overall* assignments of the relevant affordances to agents, and *comparative* assessments of such overall assignments. Both of them are to be used in certain social situations of interest including possibly as criteria for general assessments of *social progress* at large.

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As a matter of fact, in the extant literature sufficientarian views and rules and the underlying principles and motivations are discussed and scrutinized at length from different perspectives. But more often than not advocates of sufficientarianism insist on the need to rely on *absolute* as opposed to *comparative* judgements: a theoretical stance that suggests precisely a view of the relevant rankings as a derivative notion of previously established *ratings,* as suggested above [1]. Yet, while characterizations of several versions of sufficientarian rules are also available, such characterizations are typically focussed on sufficientarian *rankings* as opposed to ratings (more details on those two somewhat contrasting approaches to sufficientarianism will be presented below in Section 2).

The general aim of the present work is to provide a quite comprehensive study of sufficientarian principles starting on the contrary from *sufficientarian rating rules*, and relying on them in order to introduce sufficientarian ranking rules as a *derivative* notion of the former. Incidentally, such an approach may also help bridging the two strands of literature mentioned above.

Indeed, coming back to the core sufficientarian principle requiring that 'every one should have enough' and parsing that statement accurately, it seems to be quite clear that the basic components of any sufficientarian rule are ultimately a *common* language and framework enabling the expression of *judgements* involving a population of *agents* (the set of relevant 'individuals'), and providing for each one of them a 'yes/no' answer to the following question: 'does this particular agent *have enough?'* .

Obviously, any such answer requires in turn, as an indispensable input, answers to *two* further underlying questions, namely:

(I) 'Enough of *what*?', first and foremost, and then (II) 'What is *enough?*'.

Concerning question (I), it is broadly speaking *affordances* -resulting in *access to achievements* of some sorts- that are to be considered here when defining *assignments* to agents. But then, again, what kinds of affordances/achievements precisely? Several distinct proposals have been advanced in the literature, including *welfare levels*, *income levels, consumption bundles of perfectly divisible private goods*, *capabilities* as subsets of a suitably defined space of *functionings,* and occasionally proposals to adjoin *burdens* to the family of relevant affordances.

The present work relies heavily on *a specific definition of the relevant affordance/achievement space as the (finite) set X of all possible capability-types (i.e., combinations of levels of a finite family of affordances/achievements, each one of them being represented by some finite set of linearly ordered levels* at which it can be possibly made available*).* We also rely on the working assumption that *all the affordances/achievements that are represented in X, and their levels, are both observable and verifiable characteristics.*

While such a finiteness assumption is indisputably quite strong, it is arguably the case that it is also very much consistent with any approach to sufficientarian rules which is seriously concerned

[1] The distinction/opposition between ratings and rankings, especially when it comes to aggregation problems, has by now a long history which goes back at least as far as Huntington (1938) and has been recently revitalized, mostly as a result of the work of Balinski and Laraki (2007, 2011, 2014).

with their possible practical applications. Anyway, *it should be emphasized at the outset that such a formulation of the affordance/achievement space plays a pivotal role in this paper*.

Concerning question (II), establishing what is 'enough' clearly amounts to defining some *threshold* or *threshold system* on the relevant affordance/achievement space as provided by the answer to question (I). Once such a threshold system has been properly specified, a *benchmark* is available to form and express the required *judgments* and related binary 1/0 ratings of individual assignments of affordances/achievements. As a result, we have precisely a basic sufficientarian *benchmark-based rating* rule that consists in a certain type of function that takes individual affordance/achievement assignments to agents as inputs and returns a list of 1/0 ratings, one for each *individual* assignment, as output. Indeed, such a basic sufficientarian *rating* rule provides immediately a most natural *simple* 1/0 *rating of assignments* themselves by attributing rate 1 precisely to those assignments which obtain rate 1 for *each one* of their components, i.e., individual assignments to agents. A supplementary, refined *sufficiency-count rating* rule is also immediately obtained by attaching to each affordance/achievement assignment the rational number given by the *ratio* between the number of agents with a 1-rated individual assignment and the number of agents of the entire population under consideration. Of course, both a simple sufficientarian ranking and a sufficiency-count ranking, respectively, can be immediately defined relying on the aforementioned pair of sufficientarian ratings. It should also be noticed, however, that there are not only critics but also advocates of a sufficientarian view who, being committed to using sufficientarian rankings as a guide to redistributive policies, reject or regard anyway as highly disputable or simply implausible any use of sufficiency-count ratings and/or their induced rankings in the latter capacity, that they clearly see as a crucial one (see, e.g., Casal (2007) and Huseby (2020), respectively).

More precisely, for any population $[n] := \{1, ..., n\}$ of n agents, once an explicitly defined affordance/achievement space $\mathbf{A}$ endowed with some structure is in place, such a description of basic sufficientarian rating rules as a particular subclass of functions $f : \mathbf{A}^n \longrightarrow \{0,1\}^n$ makes it possible (at least in principle) to characterize them through properties that rely on the very structure of $\mathbf{A}$, with *no mention whatsoever of thresholds.* And that is indeed the case when one picks our finite capability-type space $\mathbf{X}$ as the relevant space, considers binary rating functions $f : \mathbf{X}^n \longrightarrow \{0,1\}^n$ on that space (which we denote here, following Balinski and Laraki (2011) as *binary grading functions (BGFs)),* defines a *threshold system* of $\mathbf{X}$ as a set of capability-types or vectors of $\mathbf{X}$ which are *pairwise incomparable* w.r.t. its component-wise order (i.e., form an *antichain* of that order), and relies on threshold systems thus defined to define in turn (binary) *sufficientarian grading rules* (or, equivalently, *basic sufficientarian rating rules*: henceforth, we shall use those two terms as synonyms) as that particular subclass of *BGFs* which satisfy the following

(Sufficientarian BGF property) there exists a *threshold system*

$$\mathcal{X}^* := \{\mathbf{x}_1^*, ..., \mathbf{x}_k^*\} \subseteq \mathbf{X}$$

such that, for any capability-type component $\mathbf{x}_i$ of any capability-type assignment $\mathbf{x}$ in $\mathbf{X}^n$, f attaches grade/rate 1 to $\mathbf{x}_i$ if and only if $\mathbf{x}_j^* \leqslant \mathbf{x}_i$ for some threshold $\mathbf{x}_j^*$ of the given threshold system (namely, if and only if there exists *at least one* capability-type $\mathbf{x}_j^*$ of the given threshold system such that $\mathbf{x}_i$ either exceeds or is equal to $\mathbf{x}_j^*$).

Specifically, one of the key results of the present work establishes that a BGF $f : \mathbf{X}^n \longrightarrow \{0,1\}^n$ is a (binary) *sufficientarian grading rule* if and only if it satisfies three quite natural and mutually independent properties namely:

- *Isotony* (higher capability-levels are *never* conducive to lower grades/rates),
- *Separability* (for any possible assignment of capability-types the grade/rate of any individual capability-type is not affected by other individual capability-types of the same assignment, or to put it otherwise the capabilities to be considered are indeed *individual* capabilities),
- *Symmetry (*for any possible assignment of capability-types, the grade/rate of any individual capability-type is independent of the agent it is assigned to, i.e., the threshold system is a *universal* one).

Moreover, characterizations of both the two-indifference-class or *simple sufficientarian total preorder* and the *sufficiency-count total preorder* induced by a basic sufficientarian rating rule are also provided. Thus, precisely as claimed above, we do obtain in fact *characterizations of basic sufficientarian rating rules (and of the simple and sufficiency-count sufficientarian ranking rules they induce) that dispense entirely with any single property referring to thresholds either explicitly or implicitly***.** It should be remarked that this feature of such characterizations puts them apart from almost any previous characterization of sufficientarian (ranking) rules the authors are aware of (see, e.g., Alcantud, Mariotti and Veneziani (2022), Bossert, Cato and Kamaga (2022, 2023), Adler, Bossert, Cato and Kamaga (2025, 2026), Nakada and Sakamoto (2024)) with *a single partial exception* concerning the *special* case of *sufficiency-count rankings* as characterized in Chambers and Ye (2024), and its further restriction to the subclass of *limitarian* sufficiency-count rankings as characterized in Ferreira and Savva (2025): more on this point is to follow in Section 2 below).

It is our contention that sufficientarian rating rules and (rating-based) ranking rules thus characterized offer a sound *minimal common core* for any sort of *sufficientarian stance*. Arguably, a sufficientarian ranking (or rating) rule is meant to establish whether the actual assignment of affordances/achievements to agents *does* satisfy the appropriate sufficientarian benchmark, and if that is *not* the case it simply signals that *some* remedial action or policy should be considered and implemented. Such remedial policies should be carefully designed according to distributive criteria that *may* possibly, but *need not,* rely in turn on sufficientarian rather than, say, some sort (or mixture) of (generalized) egalitarian and/or utilitarian criteria.

However, such an understanding is definitely *not* the prevailing attitude among proponents of sufficientarian principles (more details on that and on what follows in Section 2 below). As a matter of fact, a large part of the proponents of a sufficientarian view subscribe to the so-called *Negative Thesis* ('no redistribution needed among agents whose individual assignments are located *either* *above or on* the sufficiency-threshold'). And virtually *all of them* regard as an essential part of a sufficientarian view acceptance of the so-called *Positive Thesis* ('remedial policies must *prioritize* improvements for agents whose individual assignments are located *below* the sufficiency-threshold'). Furthermore, *many of them* also insist on redistributive policies that *the more* prioritize improvements for the agents whose individual assignments are located *below* the sufficiency-threshold, *the farther* their individual assignments are located from that threshold. In other words, such authors advocate a *strong* version of sufficientarianism, invoking some sort of *sufficientarian rankings as pivotal criteria for the required remedial policies* in order to ensure that the latter aim at some kind of *insufficiency minimization* (see, e.g., Huseby (2020), Timmer (2022)). But then, such a strong version of sufficientarianism requires by definition sufficientarian *rankings* that are *both* (much) more refined than the two-class simple sufficientarian ranking *and* arguably *different* from the *sufficiency-count ranking,* which as mentioned above is openly rejected as a sound guidance for redistributive policies by many authors, including some advocates of sufficientarian principles. In particular, insufficiency minimization apparently requires definition of a suitable metric over the space $\mathbf{A}^n$ of affordance/achievement assignments (a definition which incidentally, and apart from any other consideration, sufficiency-count strictly speaking does *not* provide[2]). And, again, a promising and natural way to obtain such a sound 'sufficientarian' metric is to rely on the *metric* structure of the affordance/achievement space $\mathbf{A}$ itself (if any such metric is available) and on its *threshold system* (once it has been specified). The present paper addresses that issue precisely in that manner, proceeding by *two steps:*

Step 1: we rely on the natural, 'intrinsic' metric structure of finite capability-type space $\mathbf{X}$ as endowed with a fixed threshold system $\mathcal{X}^* := \{\mathbf{x}_1^*, ..., \mathbf{x}_k^*\}$ in order to compute the distance of any capability-type from that fixed threshold system. Such a metric arises immediately from the very structure of $\mathbf{X}$ which is by definition a *finite* cartesian product of *finite* linearly ordered sets. Indeed, $\mathbf{X}$ itself can be endowed with the partial order $\leqslant$ induced component-wise by its linear orders, and it can be easily checked that for every pair of capability-types $\mathbf{x}, \mathbf{y}$ of $\mathbf{X}$ both their least upper-bound or *join* $\mathbf{x} \vee \mathbf{y}$ and their greatest lower-bound or *meet* $\mathbf{x} \wedge \mathbf{y}$ are well-defined. Therefore, $(\mathbf{X}, \leqslant)$ is also a lattice $(\mathbf{X}, \vee, \wedge)$ such that $\mathbf{x} \leqslant \mathbf{y}$ holds if and only if $\mathbf{x} \vee \mathbf{y} = \mathbf{y}$ or equivalently $\mathbf{x} \wedge \mathbf{y} = \mathbf{x}$. Moreover, $(\mathbf{X}, \leqslant)$ is *bounded* by construction, i.e., it is endowed with both a *maximum* and a *minimum,* and as a product of linear orders it is also *distributive,* i.e., $\mathbf{x} \vee (\mathbf{y} \wedge \mathbf{z}) = (\mathbf{x} \vee \mathbf{y}) \wedge (\mathbf{x} \vee \mathbf{z})$, or equivalently $\mathbf{x} \wedge (\mathbf{y} \vee \mathbf{z}) = (\mathbf{x} \wedge \mathbf{y}) \vee (\mathbf{x} \wedge \mathbf{z})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$. But then, it can be shown that: (a) for any $\mathbf{x}, \mathbf{y}$ of $\mathbf{X}$ the *length* $l([\mathbf{x} \wedge \mathbf{y}, \mathbf{x} \vee \mathbf{y}])$ of *interval* $[\mathbf{x} \wedge \mathbf{y}, \mathbf{x} \vee \mathbf{y}] := \{\mathbf{z} \in \mathbf{X} \colon \mathbf{x} \wedge \mathbf{y} \leqslant \mathbf{z} \leqslant \mathbf{x} \vee \mathbf{y}\}$, namely

[2] To see this, observe that for any assignment $\mathbf{a} = (\mathbf{a}_1, ..., \mathbf{a}_n)$ in $\mathbf{A}^n$ and any nontrivial permutation σ of $\{1, ..., n\}$, the resulting permuted assignment $\mathbf{a}_\sigma := (\mathbf{a}_{\sigma(1)}, ..., \mathbf{a}_{\sigma(n)})$ is such that $|\{i \in \{1, ..., n\} : f_i(\mathbf{a}) = 1\}| = |\{i \in \{1, ..., n\} : f_i(\mathbf{a}_\sigma) = 1\}|$ yet, by construction, $\mathbf{a} \neq \mathbf{a}_\sigma$. Thus, the function δ^{SC} on $\mathbf{A}^n \times \mathbf{A}^n$ defined by the rule $\delta^{SC}(\mathbf{a}, \mathbf{b}) := |(|\{i \in \{1, ..., n\} : f_i(\mathbf{a}) = 1\}| - |\{i \in \{1, ..., n\} : f_i(\mathbf{b}) = 1\}|$ does not satisfy the 'Identity of Indiscernibles' condition of metrics (the other conditions to be satisfied by metrics being of course non-negativity, identity recognition, symmetry, and triangular inequality which are indeed satisfied by δ^{SC}. It follows that δ^{SC} is *not* a metric (but, rather, just a *pseudometric*) on $\mathbf{A}^n$.

$k-1$ where k is the size of any maximal chain (or linearly ordered subset) included in that interval, is indeed a *uniquely defined* non-negative integer number, and (b) the function $d : \mathbf{X} \times \mathbf{X} \longrightarrow \mathbb{Z}_+$, defined by the rule $d(\mathbf{x}, \mathbf{y}) := l([\mathbf{x} \wedge \mathbf{y}, \mathbf{x} \vee \mathbf{y}])$ for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, is in fact a well-defined *metric* (see also Section 3.3, and Barbut and Monjardet (1970), for the relevant details). It follows that, relying on the arbitrarily fixed threshold system (or antichain) $\mathcal{X}^*$ of $\mathbf{X}$, it is possible to attach a distance $d^*(\mathbf{x}, \mathcal{X}^*)$ from threshold system $\mathcal{X}^*$ to any capability-type $\mathbf{x}$ in $\mathbf{X}$, defined as the *minimum* $d-$distance of $\mathbf{x}$ from a capability-type $\mathbf{x}_j^*$ of $\mathcal{X}^*$.

Step 2: Moving now to an *entire* capability-type assignment $(\mathbf{x}_1, ..., \mathbf{x}_n)$ in $\mathbf{X}^n$ we can uniquely attach to that assignment the non-negative integer vector $(d^*(\mathbf{x}_1, \mathcal{X}^*), ..., d^*(\mathbf{x}_n, \mathcal{X}^*)$ of the respective extended d-distances of its individual capability-types from threshold system $\mathcal{X}^*$. Of course, we are interested precisely in the distance of each capability-type assignment $(\mathbf{x}_1, ..., \mathbf{x}_n)$ from threshold system $\mathcal{X}^*$ and in the ranking (i.e., total preorder) over $\mathbf{X}^n$ induced by such distances. To compute the latter distances of capability-type *assignments* from threshold system $\mathcal{X}^*$we start precisely from the vector $(d^*(\mathbf{x}_1, \mathcal{X}^*), ..., d^*(\mathbf{x}_n, \mathcal{X}^*))$ of (extended) distances from $\mathcal{X}^*$ of the individual capability-types of such assignments, and proceed to an aggregation of the components of that distance-vector in order to obtain a *single 'summary' distance* of the entire assignment from $\mathcal{X}^*$ to be *minimized.* Such an aggregation can be made in several ways: we focus on taking *the sum* (or equivalently *the average*) or *the lexicographic maximum* (or *leximax*) of the distances of individual assignments from $\mathcal{X}^*$, respectively, and provide a characterization of the two *total preorders* they induce over $\mathbf{X}^n$.

Thus the present work also provides *a characterization of two sufficientarian ranking rules (defined through basic sufficientarian rating rules) that might be deployed as insufficiency-minimization prioritizing criteria for policy formulation or assessment.*

And finally, a further crucial issue that is rarely raised but lurks behind *any* serious attempt to advocate *any* version of *a sufficientarian stance* is also addressed here: namely, *selection of the threshold system* itself. Indeed, the basic sufficientarian rating rules (and the sufficientarian ranking rules based upon them characterized in the present paper, or for that matter in other works) amount in fact to an entire *family of rules which is parameterized by the class of threshold systems* as defined above. But then, it follows that in actual practice any conceivable attempt to adopt and implement a sufficientarian rule requires first and foremost the *identification* and *selection* of a *single, specific* universal threshold system. And, arguably, it also follows that a full-fledged formulation of any version of a sufficientarian stance should include an explicit discussion and presentation of some well-behaved *protocol* to be adopted by the relevant *deliberative bodies* in order to select the required specific threshold system. Yet, to the best of the authors' knowledge, the extant literature on sufficientarianism is remarkably elusive when it comes to the issue of threshold identification, evoking sometimes the pivotal role of an 'impartial observer or spectator', or invoking some general criteria to be used in actual practice to make sure the threshold is properly adapted to specific features of the population of agents under consideration, and only occasionally suggesting the opportunity of a collective choice of the threshold by means of 'fair' democratic procedures (see, e.g., Crisp (2003), Hassoun (2021), Timmer (2022), respectively).

The present paper addresses that open issue from a plain mechanism design perspective, *establishing the existence of inclusive and unanimity-respecting opinion aggregation rules (including inclusive quorum systems and the simple majority rule) that are also strategy-proof on a large and 'natural' domain of single-peaked preferences over threshold systems. Each one of such opinion aggregation rules can work as the key component of a protocol to be used in order to select one specific sufficientarian grading rule*[3] *by choosing its characteristic threshold system* [4].

Summing up, the main contributions of the present paper can be described by the following four points.

- *(i) Definition of the affordance/achievement space as a capability-type space given by a finite product of finite linearly ordered sets;*
- *(ii) Characterizations of (binary) sufficientarian grading rules and of the simple and sufficiency-count sufficientarian ranking rules they induce, with no explicit or implicit reference to thresholds*;
- *(iii) Characterizations of two sufficiency-gap ranking rules (the min-average and min-leximax sufficiency-gap rules, whose definitions actually rely on sufficientarian grading rules), to be possibly deployed as insufficiency-minimization criteria in the design and assessment of remedial policies*;
- *(iv) A mechanism-design-theoretic possibility result, establishing the existence of anonymous, inclusive, unanimity-respecting selection protocols for threshold systems that also enjoy a quite robust strategy-proofness property, and can be effectively used in order to select a specific sufficientarian rating and/or ranking rule thanks to the one-to-one correspondence between threshold systems and sufficientarian grading rules.*

The relevance and significance of those four points will be further clarified in the next section by discussing their relationships to the previous literature on sufficientarianism.

The rest of the paper is organized as follows. Section 2 provides an extensive yet selective review of the literature on sufficientarianism whose main aim is to help the reader to locate and appreciate the marginal contribution of the present work, and its underlying motivation and structure. Section 3 first introduces the basic notation and definitions of the model, and characterizes (binary) sufficientarian grading rules (or basic sufficientarian rating rules), defined as a subclass of binary grading rules over capability-type assignments. Then, some sufficientarian rankings are introduced and characterized, including the sufficiency-count ranking, the min-average sufficiency-gap and the

[3]Or perhaps more than one, if and when required (more on that topic in Section 2).

[4]That result is obtained as a joint corollary to previous results in Savaglio, Vannucci (2019) and Vannucci (2019) as combined with a classic theorem due to Dilworth (1960) establishing that the set of antichains of a finite partially ordered set is a distributive lattice under a very natural order.

min-leximax sufficiency-gap ranking. Section 4 addresses the issue concerning the definition and existence of well-behaved protocols of threshold-selection. Section 5 offers some concluding remarks and suggests a few possible extensions of binary grading functions to richer capability-type spaces as a topic for future research. All the main proofs are collected in an Appendix.

2 Related literature

Remarkably, recent contributions on sufficientarian principles and rules come from the perspectives of quite distinct subdisciplines including social choice theory, normative economics, political philosophy and social ethics (see, e.g., Crisp (2003), Roemer (2004), Benbaji (2005, 2006), Huseby (2010, 2019, 2020), Axelsen and Nielsen (2015), Hirose (2016), Nielsen (2019, 2019b), Timmer (2021, 2022), Alcantud, Mariotti and Veneziani (2022), Bossert, Cato and Kamaga (2022, 2023), Chambers and Ye (2024), Nakada and Sakamoto (2024), Harting (2024), Adler, Bossert, Cato and Kamaga (2025)).

Accordingly, several distinct perspectives and understandings on the aim and scope of sufficientarian principles are proposed and subscribed to by different authors. Some of them regard sufficientarian principles as a comprehensive theory of distributive justice (or even, more generally, of social ethics) to be contrasted with rival approaches such as, say, (generalized) egalitarianism, (generalized) utilitarianism, or prioritarianism, i.e., a variety of generalized utilitarianism which confers some priority to the worse-off: see, e.g., Crisp (2003), Benbaji (2005, 2006), Huseby (2010, 2020), Axelsen and Nielsen (2015), Nielsen and Axelsen (2017), Nielsen (2019, 2019b), Herlitz (2019), Timmer (2021, 2022), and Harting (2024) for a somewhat less demanding 'hybrid' stance advocating the combination of sufficientarian and 'relational egalitarian' distributive criteria. As a result, such contributions share two key features: (i) they typically take for granted that sufficientarian rules are to be used and validated *both* as a general and convenient benchmarking device *and* as the basic guidance to remedial (re)distributive policies whenever actual affordance/achievement assignments fail to satisfy the appropriate sufficientarian benchmarks, and (ii) the discussions and articulations of sufficientarian principles they propose are significantly shaped by the intent to defend sufficientarianism against the criticisms advanced by supporters of alternative principles of distributive justice, including egalitarianism, prioritarianism and other versions of generalized utilitarianism, or mixtures of sufficientarianism and other distributive principles (see, e.g., Arneson (2005), Casal (2007) and Cohen (2011), Parfit (1997), Brown (2005), Shields (2012) and Knight (2022), respectively). Moreover, some authors that are also prepared to consider sufficientarianism as a general principle of social and/or population ethics propose generalized characterizations of sufficientarian rankings in a *variable population* setting (see, e.g., Bossert, Cato and Kamaga (2022, 2023), and Hirose (2016) for an early suggestion in that vein).

Generally speaking, the body of relevant literature is by now considerable if not vast, and the range of both issues of contention and analitical methods deployed is also considerable. Clearly, this is not the place for a comprehensive review or discussion of all of those issues. Moreover, the present paper is only concerned with a detailed analysis and characterization of sufficientarian rules as rating and ranking criteria for capability-assignments without any underlying assumption about their status as a (comprehensive or partial) theory of distributive justice, or even as a prominent tool in guiding formation, selection and assessment of redistributive policies.

Therefore, we shall mostly focus on those contributions that, either raising or reacting to specific challenges to sufficientarian principles, have been shaping some widely held understandings on the possibly critical blind spots or open issues of any sufficientarian view, to which the present article is meant to bring some clarification, or contribute a solution.

The most immediate source of inspiration for the literature on sufficientarian principles comes apparently from Frankfurt (1987, 1997a, 1997b, 2015) who advances what he calls 'the doctrine of sufficiency' as an alternative to the notion that 'economic equality' should be treated as a *basic* normative distributive principle. His argument relies heavily on the view (previously alluded to in the Introduction), that any *basic* normative principle should make use of *absolute* judgements as opposed to *comparative* ones. Moreover, Frankfurt maintains that such a sufficientarian alternative lends support to policies focussing on welfare enhancement for those who *have not enough*, including of course poverty abatement. Such policies *may* also, *but* at least in principle *need not,* include economic inequality control and mitigation. Notice that Frankfurt's indictment only concerns economic equality as a *fundamental* principle of social ethics, but is consistent with advocacy of inequality abatement policies as an *instrument* to pursue other goals, including possibly intertemporal allocative *efficiency* to the extent that economic inequality is regarded as a *negative externality* (as explicitly suggested, e.g., by Stiglitz (2012), or Nyborg Støstad and Cowell (2024)). More recently, sufficientarianism has also been advocated as an 'indispensable' standpoint to cope with the ethical issues concerning the proper treatment of the very badly off, whether or not due to their own choices[5] (see Herlitz (2019)). Furthermore, it is also worth mentioning that a sufficientarian stance of some sort is arguably gaining credit as a promising forceful or even ultimately indispensable framework to address the serious unemployment problems to be possibly expected in the near future, as a result of the massive diffusion of new information technologies (including AI and its burgeoning applications).

In order to gain and take advantage of a broader perspective on sufficientarian views, it should be noticed, however, that there are at least *two major streams* of earlier contributions pointing to distribution rules which embody some version of sufficientarian principles. One of them is the advocacy of a *universal basic income (UBI)* which can be traced back to Russell (1918), and has been revived in several versions and under several labels in the last few decades (see, e.g., Van Parijs (1995)[6], Widerquist (2024) among many others). The other one originates from the famous

[5]It should be remarked that even '*luck egalitarianism'* , as defined and advocated by Cohen (2011) and others, disavows egalitarian redress against disadvantages that are to be classified as a result of deliberate choices of an agent, as opposed to just 'bad luck'.

[6]It should be emphasized that Van Parijs's own favored principle of distributive justice in order to achieve 'real

distribution rule '*from each according to their ability, to each according to their needs*' due to Marx (1875), who envisages that distribution rule as the one that would/should prevail within what he calls 'the realm of freedom' (namely, the advanced or mature 'communist' stage of a 'socialist' society, that is indeed supposed to achieve 'real freedom' for all).

It is undoubtedly the case that the foregoing distinct strands of *'sufficientarian'* principles exhibit some remarkable mutual differences. For instance, it is quite clear that 'having enough' in Frankfurt's sense does imply enjoying a 'non-poor' status. But it is not entirely clear whether or not the reverse is also the case, though Frankfurt seems to be willing to distinguish between 'having enough' and 'being non-poor'. By contrast, it is arguably quite clear that Marx's 'realm of freedom' is expected to admit (and in fact largely rely upon) his 'communist' distribution rule precisely because it is supposed to afford a considerable degree of diffuse affluence, due to the implicit assumption of massively improved technological basis and productive capabilities[7]. In other terms, the 'needs' that are mentioned in Marx's rule are *not* just the needs that must be satisfied in order to escape the 'poor' status: being 'non-poor' does not imply 'having enough' or full-fledged 'need-satisfaction' in Marx's own sense (that is in fact typically connected to achievement of conditions enabling a 'flourishing of human personality'[8]). On the other hand, and contrary to what is sometimes wrongly taken for granted, Marx's 'communist' distribution rule does *not* require *at all* unconditional 'economic equality', precisely as Frankfurt's 'doctrine of sufficiency'. Furthermore, both such Marxian distribution rule and Frankfurt's sufficiency standard are meant to provide first and foremost *benchmarks* to support (binary) *ratings* [9] of social states, which in turn *may but need not be used* to define *rankings* for guidance and assessment of remedial policies. In fact, Marx (1875) simply ignores the whole issue of remedial policies while Frankfurt (1987) does take into consideration remedial policies to deal with social states that *fail* to satisfy the *sufficiency benchmark* and *rankings* or criteria to shape such policies, but firmly distinguishes such rankings from the sufficiency benchmark itself.

freedom for all' can be described as 'UBI maximization subject to undominated diversity' (see Section 3 for a short presentation of the latter notion, than can be regarded as a generalized egalitarian principle). Thus, strictly speaking Van Parijs does not advocate a proper sufficientarian principle, but rather a sufficientarian rule of sorts subject to a generalized egalitarian constraint. It should be noticed, however, that the latter constraint might also be construed as a clause requiring that everyone is granted 'enough real freedom'.

[7]Which does not necessarily mean the achievement of some utopian 'state of abundance', however. An extra-bonus of our model based on finite capability-type spaces is that it makes crystal clear that achievement of full sufficientarian capability-type assignments does not amount to, or require, a state of 'abundance' in the proper sense of 'lack of scarcity'. That hypothetical, and arguably utopian, state of affairs would presumably make the full achievement of sufficientarian goals very easy or even trivial, but it is *not at all* implied or required by the latter.

[8]Notice, however, that Marx does *not* single out any particular set of activities or behaviours as either 'typycally human' or intrinsically superior to others. Hence, Marx's distribution rule for the 'realm of freedom' does *not* imply or rely on a '*perfectionist*' ethical stance, and the same arguably holds for most current formulations of sufficientarianism.

[9]It is worth recalling that, in Marx's own view, implementation of his 'communist' distribution rule works as the underlying benchmark of an overarching, grand binary rating of social progress in human societies. Indeed, Marx claims that widespread adoption of such a distribution rule is the hallmark of nothing less than 'the end of human prehistory' and, accordingly, the *beginning* of human history (properly so said), *not* its *end*.

Coming to *UBI advocacy*, its key distinctive features are precisely its definition as a certain fixed amount of units of a convenient medium of exchange and payment (e.g., money), and its universality or unconditional nature. Its cash–like unidimensional denomination is clearly at variance with the more nuanced and possibly multidimensional character of the capabilities everyone should have *enough* access to according to Frankfurt's and Marx's distribution rules [10]. And its unconditional nature would also be possibly inconsistent with any interpretation of Marx's rule which insists that its first component (*'from each according to their ability')* amounts in fact to a conditionality clause (an interpretation that is in our view scarcely compelling, but at the same time not unconceivable[11]). Concerning the relationship of UBI proposals to poverty and sufficiency thresholds or systems of thresholds, it is arguably the case that its nature depends again on the specific version of UBI one has in mind. Under some of its versions, UBI amounts to a *sufficiency guarantee* that goes possibly *far beyond* the minimum level required to escape poverty (see e.g. Van Parijs (1995)). But there are versions of UBI proposals that insist to keep it low enough to preserve adequately strong work incentives (perhaps even *close* to the relevant *minimum no-poverty threshold,* as it *is* arguably the case for the original proposal advanced by Russell (1918)), and even further minimalist proposals that also allow for an UBI whose level is located *below the poverty threshold* (see e.g. Widerquist (2024) for a comprehensive discussion of several proposed versions of UBI).

As mentioned above, the recent literature on sufficientarianism itself is by now quite extensive and focussed on a broad spectrum of issues ranging from interpretations of its core-notions to formulation of an appropriate axiomatization, with its underlying framework. To begin with, there any many different views concerning the appropriate definition of the underlying affordance/achievement space. Some authors assume it consists of unidimensional *welfare levels* under various interpretations including possibly as lifetime well-being indicators (e.g., Alcantud, Mariotti and Veneziani (2022), Bossert, Cato and Kamaga (2022, 2023), Hirose (2016), Huseby (2020)), or even just *income levels* (e.g., Frankfurt (1987), Widerquist (2010)). Others insist on a *multidimensional* representation of the relevant space and propose to focus either on *capabilities* as subsets of a suitably defined space of functionings[12] (e.g., Axelsen and Nielsen (2015), Nielsen and Axelsen (2017)) or on *prospects* consisting of *state-contingent welfare levels* (see Adler, Bossert, Cato and Kamaga (2025, 2026)). Thus, both in the unidimensional and the multidimensional case there are both proposals of affordance/achievement spaces consisting of variables that are in principle observable and verifiable (money units, capabilities) and proposals of affordance/achievement spaces that on the contrary include variables whose values are *private information* of the agents and typically can only be accessed to by an appropriate *elicitation* protocol (welfare levels, prospects).

[10]It should be mentioned however that, apparently, Frankfurt does not rule out use of money-denominated thresholds (see e.g. Frankfurt (1987), p. 37).

[11]But see the last section of this paper for more observations on that point, and on the way 'burdens' might be taken into account through a straightforward extension of our capability-type space.

[12]Along the lines of Sen (1985,1997). Notice that capability-spaces are inherently multidimensional, and functionings (as opposed to welfare levels or utilities) are typically meant to be *observable and verifiable.*

A remarkable exception is provided by Chambers and Ye (2024) who work with three sorts of affordance/achievement spaces and their respective sufficiency-sets, namely (i) a set A of *indexes of unspecified appropriate characteristics* with a *sufficiency-set* S consisting in an arbitrary subset of A; (ii) a standard multidimensional commodity space A' endowed with a *preorder* $\preccurlyeq$ (i.e., a reflexive and transitive binary relation), with a sufficiency-set S' given by an *upward closed* subset of A' (i.e., a *preorder filter* of $(A', \preccurlyeq)$) and (iii) a partially ordered space A'' with a *partial order* $\leqslant$ (i.e., a reflexive, transitive and antisymmetric binary relation) that is also a *meet-semilattice* (i.e., such that the greatest lower bound is well-defined for any pair of elements of A''), with a sufficiency-set S'' given by an upward closed subset of A'' which is also meet-closed (i.e., a *latticial* order filter of $(A'', \leqslant)$). It is worth noticing that all of those three specifications of the affordance/space proposed in Chambers and Ye's contribution formally qualify as *generalizations* of the space of capacity-types introduced and deployed in the present paper.

Yet, none of such sufficiency-sets proposed in Chambers and Ye (2024) do qualify as a sound generalization of the sufficiency-sets induced by the sufficientarian threshold systems we define and deploy in the present paper, whose key property here is that they are in a one-to-one correspondence with basic sufficientarian ratings and the sufficientarian rankings they induce.

That is so because a threshold system of a finite capability-type space consists of the *minimal elements* of an upward closed subset, i.e., of an *order filter* of the (ordered) set of capability-types: thus, the sufficiency-set attached to an arbitrary threshold system is an order filter. However, order filters cannot be defined in an unstructured set such as A. Thus, generally speaking, a sufficiency-set S of A is *not* an order filter, its minimal elements are not defined, and S has no threshold system whatsoever attached to itself. Preorder filters can be defined in $(A', \preccurlyeq)$ and a sufficiency-set S' is indeed a preorder filter[13]. However, without some further discreteness condition on A'[14], S' may well have *no threshold or threshold system* of its own at all. And finally, the order filters of semilattice $(A'', \leqslant)$ are by construction *principal,* i.e. they have invariably exactly *one* minimal element, hence they only admit a *special* and *trivial* sort of threshold system consisting of a *single* threshold[15].

[13]A preorder filter is defined in the obvious way as a subset which is upward closed with respect to the given preorder (see e.g. Savaglio and Vannucci (2007) for the introduction of preorder filters in a study of opportunity inequality).

[14]Such as No Bounded Infinite Chain or the Descending Chain Condition (i.e., no infinite descending chain). Without any such discreteness conditions $(A', \preccurlyeq)$ may well be in fact a collection of open half lines in $\mathbb{R}^m_+$ endowed with the restriction of the 'natural' partial order $\leq$ of $\mathbb{R}^m_+$ to that very collection. In that case any subcollection of the foregoing collection would be an order filter of $(A', \preccurlyeq)$ with no minimal points, hence with no threshold or threshold system attached to it.

[15]Thus, it is not surprising that, in order to characterize sufficiency-count rankings on very general spaces without relying on sufficientarian binary grading rules, Chambers and Ye (2024) have to add a further axiom to monotonicity or isotony, separability and symmetry. Yet, at the same time, the supplementary axiom employed, labeled '*Sufficientarian Judgement'*, is not specifically related to thresholds. That it so because it only requires that if any uniform assignment is worsened by a change of the individual assignment of a single agent then any further change of the individual assignment of another single agent will result in an assignment which is never a strict improvement of

Finally, it should also be recalled here that, as previously mentioned in the Introduction, there are also some suggestions to the effect that the relevant affordance space should include both achievements *and burdens.* Those suggestions come from both advocates and critics of sufficientarian views and rules such as Nielsen (2019) and Knight (2022), respectively, who also concur in regarding a proper treatment of burdens as a major challenge for sufficientarian views. A proposal concerning a possible way to extend our capability-type spaces in order to accommodate burdens will be discussed in the final section of this paper.

As it should be expected, a significant amount of discussion and controversy in the literature concerns the sufficiency-threshold itself as considered from several perspectives: its role and meaning, the precise formulation and characterization of the sufficientarian rules that are required to embody the sufficiency-threshold, the role and aim of such sufficientarian rules and, to a lesser extent, how sufficiency-thresholds could and/or should be determined.

Concerning the role of the sufficiency-threshold, its ostensible effect is to obtain a bipartition of the underlying affordance/achievement space into two blocks consisting of 'sufficient' and 'insufficient' states, respectively. And that bipartition is precisely what is needed in order to state a sound representation of the main content of a full-fledged sufficientarian view in terms of what are widely regarded as its *two basic theses.* Indeed, as previously observed in the Introduction, most critics and many advocates apparently concur on the description of sufficientarianism as the view that there is a 'sufficiency threshold' in the relevant space which satisfies the following two conditions: (a) ('*Positive Thesis*') since everyone should have enough, priority must be given to those who stay below the threshold and (b) ('*Negative Thesis*') distributive considerations concerning those who stay on or above the threshold are essentially irrelevant (see Arneson (2005), Brown (2005), Casal (2007), Axelsen and Nielsen (2015), Nielsen (2019, 2019b), Knight (2022), and also Crisp (2003) where the two theses are conflated into a single '*principle of compassion*'). But then, critics insist that the existence of a threshold having such a twofold property is most counterintuitive and implausible: hence it is not clear if and how such a threshold can be successfully defined *at all.* Anyway, all of them maintain that specific 'priority' criteria -possibly egalitarian or generalized utilitarian- should replace, or at least be adjoined to, 'sufficiency' benchmarks in order to guide policies aimed at improving the access to achievements of those agents whose assignment is located *below* the threshold (see e.g. Arneson (2005) and Casal (2007), or Brown (2005) and Knight (2022), respectively).

The answers to that criticism on the part of authors who advocate some version of a sufficientarian stance clearly reflect somewhat different meanings attached to sufficiency-thresholds, and are accordingly remarkably varied. Nielsen (2019b) suggests that the sufficiency-threshold should ensure

the previous one (hence a fortiori it cannot compensate for the first change). Clearly, that is in fact the behaviour of a preorder which relies on a target subset S and ranks assignments according to the sizes of the sets of agents whose individual assignments belong to S. Nothing is said about S. For instance, it can be induced by a threshold (sufficientarian case), by a cap (limitarian case), or by a threshold and a cap (limitarian-sufficientarian case). Arguably, 'Sufficientarian Judgment' is in fact a misnomer for such a condition. 'Target Set Inclusion-Count Judgment' would make perhaps a more accurate descriptive label for it.

that *all* 'reasons of justice' are '*completely* sated' once it is reached, so that in a sense the former is to be placed so 'high' in the relevant affordance/capability space that the 'Negative Thesis' is in fact satisfied, but only *trivially* so[16]. On the contrary, some authors rather propose a considerable relaxation of the 'Negative Thesis'. In particular, Shields (2012) suggests that the sufficiency-threshold also marks a discontinuous *shift* that merely *weakens* (without cancelling) the reasons to further benefit those agents who have reached it. Benbaji (2005, 2006) proposes a *multiplicity* of ordered thresholds such that the 'Positive Thesis' applies to each one of them, with a priority that is the higher the lower their location in the affordance/achievement space: satisfaction of the 'Negative Thesis', however, is reinterpreted as indifference between the individual assignments located *between* any two thresholds (see also Casal (2007), for a similar approach which relies on just *two* ordered thresholds). In a somewhat similar vein, Timmer (2022) also allows for *several* (ordered) thresholds, claiming that 'priority' criteria favoring improvements of affordance/achievement bundles located below *some* thresholds (with a further prioritization of *lower* thresholds amongst the latter) are to be regarded as an essential component of sufficientarianism. Nakada and Sakamoto (2024) introduce and characterize *multi-threshold generalized sufficientarian rankings* where several ordered thresholds are represented by suitably defined 'discontinuity points' of the ranking. Moreover, Huseby suggests (in Huseby (2010) and in Huseby (2020), respectively) *two distinct versions* of a sufficientarian view which *both* rely on *two* (ordered) thresholds. In the first version (Huseby (2010) the upper or maximal sufficiency-threshold represents a welfare-level of '*subjective contentment'* to which both the 'Positive Thesis' and the 'Negative Thesis' do apply, while the lower or minimal sufficiency-threshold specifies a subsistence welfare level that corresponds to satisfaction of '*basic human needs'* and is only meant to signal possible priority in favor of those agents whose individual assignments are located below the maximal threshold (thus, the lower threshold works as a sort of poverty threshold, but neither the 'Positive Thesis' nor the 'Negative Thesis' are, strictly speaking, implied here). In the second version (Huseby (2020)) the 'Positive Thesis' only refers to the lower sufficiency-threshold, while the 'Negative Thesis' only refers to the upper sufficiency-threshold. Furthermore, while invoking 'prospect utilitarianism' (a certain type of utilitarian ranking relying on non-expected utility functions) as an improvement upon sufficientarian rankings, Cato and Chung (2026) argue that the best possible version of a sufficientarian ranking should rely on the sufficiency threshold and an additional upper threshold: The basic reason they advance to favor their own proposal is that the 'prospect-utilitarian' ranking is *continuous*, while the two-threshold sufficientarian ranking they consider has a *discontinuity* point at the sufficiency-threshold. It is worth noticing here that the finite capability-type space framework used in the present paper makes any such *continuity*-argument simply *irrelevant* [17].

[16]Notice that under such an interpretation the sufficiency-threshold also qualifies as an extreme, trivialized version of a *limitarian*-threshold or *cap* (namely, a 'threshold' such that any individual assignment that is located above it should be avoided because it is '*too much*': see, e.g., Ferreira and Savva (2025) for a study and characterization of *limitarian-sufficientarian* ranking rules).

[17]That is so because in any finite or more generally discrete setting, the natural topology is the discrete one, which

Arguably, such a proliferation of ‘thresholds’ within a sufficientarian framework and stance is scarcely surprising if not just unavoidable, since as mentioned above several of its proponents regard the sufficiency-threshold as the counterpart of a considerably ‘high’ standard of living: a view which in turn, at a minimum, invites a comparison with a supplementary and obviously *distinct* poverty-threshold or no-poverty-benchmark. It should also be noticed, incidentally, that a sufficientarian grading rule as defined in the present paper can be immediately deployed in poverty analysis, simply reinterpreting its threshold system as a *basic-needs* or *no-poverty-threshold*[18]. But then, one might perhaps even suggest that a key difference between the deployment of sufficientarian grading rules in sufficientarian and poverty analysis, respectively, is precisely the fact that while the latter only requires a *single* sufficientarian binary grading rule, the former actually requires *at least two of them*, and possibly more. For instance, one might also consider a further auxiliary *upper* threshold located above the sufficiency-threshold, marking its upper contour as the region of the capability-type space that is a possible prioritary target of certain strongly progressive taxation policies (thus providing a sort of ‘limitarian’ variety of the sufficientarian stance).

Concerning the actual formulation and characterization of the relevant sufficientarian rules and criteria, the bulk of the literature is most recent (see Alcantud, Mariotti and Veneziani (2022), Bossert, Cato and Kamaga (2022, 2023), Chambers and Ye (2024), Nakada and Sakamoto (2024), Ferreira and Savva (2025), Adler, Bossert, Cato and Kamaga (2025, 2026), with Roemer (2004) and Hirose (2016) as early precursors).

As it happens, and somewhat surprisingly, the sufficientarian rules or criteria that *all* of those contributions focus on, and proceed to characterize, are sufficientarian *rankings* and in particular, with the single exception of Chambers and Ye (2024), *social welfare orderings* (i.e., essentially, single-profile social welfare functions in the Bergson-Samuelson tradition: see, e.g., Gevers (1979), Roberts (1980a,1980b), d’Aspremont (1985), Moulin (1988), and some earlier related works such as Sen (1977), Hammond (1976, 1979), d’Aspremont and Gevers (1977), Deschamps and Gevers (1978)). And, as previously mentioned, almost all of those characterizations (with the single exception of Chambers and Ye (2024) as previously discussed above) include an explicit reference to an exogenously given threshold. Thus, such contributions tend to disregard the widely shared view that, as mentioned above, a sufficientarian stance should rely first and foremost on sufficientarian *ratings* of affordance/achievement assignments, and only derivatively on the rankings induced by those ratings. As previously mentioned in the Introduction, *one of the main aims of the present paper is*

forces *any* function to be continuous. Cato and Chung do acknowledge explicitly this point (see Cato and Chung (2026), note 28, p.366). However, they also apparently claim that such a consideration does only apply to sufficiency-count rankings, and not to other ‘non-headcount’ sufficientarian rankings (which include of course our sufficiency-gap rankings). But, generally speaking, the latter claim is simply wrong, since under the discrete topology any binary relation (hence in particular any preorder) is also continuous.

[18] As a matter of fact, other possible applications of binary grading functions can also be envisaged, including the analysis of *exploitation*, and *expertise*. Some more observations on that point are made in the final section of the present paper.

precisely, in that respect, to redress the state of things by taking into consideration sufficientarian rating rules. We do so by introducing (binary) sufficientarian grading rules, and characterizing them and the sufficientarian rankings they induce without any reference whatsoever to thresholds.

Furthermore, while endorsement of the sufficientarian 'Positive Thesis' as defined above invites remedial policies whenever some agents fail to reach the sufficiency-threshold, it is in fact unclear whether and to what extent sufficientarian rankings of some sort are required to be used in order to design and/or assess such policies. That is so especially because the sufficientarian rankings that have been characterized (see, e.g., Alcantud, Mariotti and Veneziani (2022), Chambers and Ye (2024)) are indeed versions of the sufficiency-count ranking, whose possible use as a pivotal policy criterion is bound to be controversial (see, e.g., Huseby (2020), among others, for an explicit, flat rejection of such an use of the sufficiency-count ranking). Thus, *the explicit introduction and characterization of sufficiency-gap rankings in the present paper fills indeed a gap in the extant literature.*

Finally, we come to a last and absolutely crucial point involving sufficiency-thresholds, and its discussion in the literature. Namely, *how sufficiency-thresholds are determined*. Clearly, that is definitely a key issue, because any conceivable application of any sufficientarian rule requires selection of (at least) *one specific threshold* (or threshold system). But then, how is that selection supposedly made? Quite remarkably, the extant literature on sufficientarianism is almost silent on that issue. As previously mentioned in the Introduction, one of the few exceptions is Crisp (2003) who explicitly invokes the pivotal role of an impartial observer or spectator in the threshold-selection process (and, more precisely, the elicitation of her sentiment of compassion as the source of her decision)[19]. In a slightly more practical if similar vein, Hassoun (2021) suggests a *'mechanism'* to select a sufficiency-threshold regarded as 'the standard necessary for living a minimally good life': such a 'mechanism' consists in fact in a set of recommendations to guide the decisions of any 'reasonable, free, caring person' in charge of the threshold-selection process. A definitely more operational approach is suggested by Timmer (2022) who advances the notion of *'political sufficientarianism'* which amounts to selection of the sufficiency-threshold by means of 'fair' democratic procedures.

As discussed in the Introduction, *the present paper addresses such important issue of threshold-selection from a mechanism-design perspective, and establishes the existence of nice and strategy-proof protocols which might be deployed in order to accomplish that task. Such a result is obtained by exploiting the structure of the capability-type space as a finite product of finite linearly ordered sets, which makes it both a finite ordered set and a finite distributive lattice*, a fact that in turn makes it possible to rely on a few previous results[20]. That is so because such a space, as a finite ordered set, has order filters (upward closed subsets) the collections of whose *minimal* elements (which may be not unique, but are of course by construction mutually incomparable or *antichains*) amount in

[19] That is of course a rather transparent reference to the approach to ethics due to Adam Smith's *Theory of Moral Sentiments* and partly to his 'immediate' precursors, Hutcheson and Hume.

[20] Mainly, Dilworth (1960), Savaglio and Vannucci (2019), and Vannucci (2019).

fact to *threshold systems*. And as a distributive lattice, it also admits nice aggregation rules with a remarkable strategy-proof property which can be used as protocols to select a unique threshold system.

Preorders on an 'opportunity space' with a single threshold defining the top-ranked indifference class, and possibly several further ranked indifference classes down below, are introduced and studied in Savaglio and Vannucci (2007) and Vannucci (2013) under the label 'filtral preorders' (since of course any such threshold induces exactly one preorder filter). However, the foregoing contributions focus on the special case of simple thresholds consisting of a *single 'point'* in a multidimensional opportunity space, which indeed correspond to the special subclass of *principal* order filters. Moreover, as already discussed in some detail above, the only affordance/achievement space introduced by Chambers and Ye (2024) that is both more general than our own capability-type space and supports a tight connection of sufficiency-sets as order filters to thresholds is in fact a meet-semilattice, and it is only *latticial* (i.e., meet-closed) order filters that are taken into consideration in the aforementioned paper. But any latticial order filter having some minimal point is a *principal* order filter, i.e., it has a *unique* minimal point hence only admits a trivial threshold system consisting of a *single* threshold.

The very same observation applies to the multidimensional thresholds considered by Nielsen and Axelsen (2017) which amount indeed to *single* threshold-points of the (multidimensional) capability space they propose to work with. General or multi-point threshold preorder or order filters are also explicitly mentioned in Savaglio and Vannucci (2007) and, in a slightly different yet obviously related 'poverty ranking' setting, in Peragine, Pittau, Savaglio and Vannucci (2021), but not studied in any detail. Thus, to the best of the authors' knowledge, sufficientarian rules with threshold systems consisting of nontrivial antichains of thresholds were apparently never considered and examined in the previous literature on sufficientarianism and related topics.

3 Model and results

In the present work sufficientarianism is described in terms of a class of *sufficientarian binary grading rules*, namely basic sufficientarian *rating* rules which amount to a special class of binary grading functions of capability-types as defined below.

3.1 Notation and basic definitions: binary grading functions and (binary) sufficientarian grading rules

Let $[n] := \{1, ..., n\}$ be a finite set of agents, and $(\mathbf{X}_i, \leqslant_i)$ with $|\mathbf{X}_i| = l_i \in \mathbb{Z}_+ \setminus \{0\}$, and $i \in [m] := \{1, ..., m\}$ the finite family of relevant 'positive' *affordances* (and related *achievements*), each one of them consisting of a finite set $\mathbf{X}_i$ of levels ordered by a linear order $\leqslant_i$ (i.e., a reflexive, connected, transitive and antisymmetric binary relation). Borrowing and adapting the terminology from Sen (1985, 1999) we denote as *capability* a set of 'positive' affordances/achievements of the capability space to be defined below. *Capabilities* and their underlying *affordances/achievements*[21] are meant to be *observable* and *verifiable* attributes of agents. The affordances/achievements we are going to consider are most typically non-rival, treated as strictly *individual* attributes[22], and an individual capability-assignment to a certain agent is the relevant *capability-type*, or simply *type*, of that agent. Accordingly, we define the *capability-type space* as $\mathbf{X} :=_{i=1}^{m} \mathbf{X}_i$, and denote the finite partially ordered capability-type space by $(\mathbf{X}, \leqslant)$, where $\leqslant :=_{i=1}^{m} \leqslant_i$[23]. Let $\mathbf{X}^n$ be the set of all conceivable capability-type-assignments to agents in $[n]$, and $\{0, 1\}$ the two relevant *grades* that may be also read as no/yes or false/true, respectively.

We refer to any $\mathbf{x}_{[n]} \in \mathbf{X}^n$ as an *assignment of* capability-types, and for any such assignment and any agent $i \in N$, we denote by $\mathbf{x}_i \in \mathbf{X}$ the individual assignment of i at $\mathbf{x}_{[n]}$. Within such a framework, the information base of sufficientarianism is then a list of judgements on the individual affordances/achievements asserting for each agent whether her/his individual assignment is *'sufficient'* or not. Any such judgement can be expressed through a *unary predicate* '*being sufficient*" that is defined on capability-types in $\mathbf{X}$, and can in fact be identified with its truth-value (either 1 if true, or 0 if false), which in the present context is in fact precisely a *grade*. Accordingly, a similar 0/1 judgment can be extended to a full *assignment* $\mathbf{x}_{[n]} \in \mathbf{X}^n$ of such capability-types relying precisely on the given list of judgements on capability-types (one for each agent $i = 1, ..., n$). Thus, the judgement '*i has enough* at capability-assignment $\mathbf{x}_N$' amounts to the equivalent judgement 'capability-type $\mathbf{x}_i$ is sufficient' and its truth-value.

Therefore, the foregoing approach results in the definition of a particular *binary grading function* (BGF) $g : \mathbf{X}^n \to \{0, 1\}^n$ that assigns to any conceivable assignment $\mathbf{x}_{[n]} \in \mathbf{X}^n$ of capability-types to the n agents the list $g(\mathbf{x}_{[n]}) = (g_1(\mathbf{x}_{[n]}), ..., g_n(\mathbf{x}_{[n]}))$ of their respective n binary values or *grades*, one such grade for every agent, that indicate whether or not the corresponding agent 'has enough' according to the capability-type assignment under consideration[24]. It should be noticed that such a binary grading function amount to a special case of a *social grading function*, a convenient tool to describe social states from a normative point of view that has been recently and successfully advocated by Balinski and Laraki (2007, 2011, 2014). A social grading function assesses each such

[21] Thus, in our model affordances/achievements are in fact the counterparts of Sen's *functionings* as constituent units of capabilities.

[22] Capabilities of a collective or even public nature that are related to access to public goods are thus essentially ignored. That simplifying move is not meant to imply that public-good-related capabilities (of which access to scientific knowledge as the output of basic scientific research is a prominent example) are irrelevant or just not amenable to treatment within our framework. On the contrary, a simple adjustment of our model can accommodate capabilities of a public nature (more on this in the final section of this paper).

[23] It should be noticed that, by construction, the basic domain of entities to be assigned to agents essentially amounts to an arbitrary (finite) partially ordered set. That is so because, though our capability space is in fact a product of linearly ordered sets, it is well known that any partially ordered set may be identified with its minimal decomposition into (disjoint) linear orders.

[24] Sufficientarian grading functions with an arbitrary (or possibly just bounded or finite) set of grades might also be considered, but will be not in the present work.

social state by a list of *grades* chosen from a bounded linearly ordered set of grades, as many grades as there are individual agents, and each grade - which may be indeed a number as in our case- is intended to assess the social state as far as the corresponding agent is concerned. The social grading function returns an aggregate grade for each social state. Thus, in our case a binary grading functions is precisely a social grading function having capability-types as states and just two ordered grades. Which is of course most appropriate since sufficientarianism, or 'the doctrine of sufficiency' as originally formulated by Frankfurt (1987), is precisely about whether or not each agent 'has enough'. Thus, it seems to be natural to restrict the possible grades of individual assignments of capability-types to the set $\{0, 1\}$ (e.g., an agent with such a capability-type 'has not enough' or 'has enough', respectively).

It should also be emphasized that a discussion of sufficientarian principles in terms of BGFs is definitely at variance with the bulk of the extant literature. Indeed, within the recent literature (see, e.g., Alcantud, Mariotti and Veneziani (2022), Bossert, Cato and Kamaga (2022, 2023) and Chambers and Ye (2024)), sufficientarianism has been examined through the properties of a binary relation on the set of social states (namely, entire affordances/achievements assignments), as opposed to an ordered bipartition of that set of social states (assumed here to be $\mathbf{X}^n$). But, as shown in the next section below, the present approach based on BGFs also makes it possible to define several natural sufficientarian (total) *preorders* on $\mathbf{X}^n$.

Finally, it must be stressed that it is by no means the case that an arbitrary binary grading function on $\mathbf{X}^n$ qualifies as a sound *sufficientarian* binary grading rule. To see this, consider the binary grading function $g^{UD} : \mathbf{X}^n \to \{0, 1\}^n$ defined as follows: for any $\mathbf{x}_{[n]} \in \mathbf{X}^n$ and $i \in [n]$,

$$\begin{aligned} g_i^{UD}(\mathbf{x}_{[n]}) &= 0 \text{ if there exists } h \in [n] \text{ such that } \mathbf{x}_i \leqslant \mathbf{x}_h \text{ and } \mathbf{x}_i \neq \mathbf{x}_h \\ &= 1 \text{ otherwise} \end{aligned}$$

Clearly, g^{UD} represents a version of the *undominated diversity (UD)* criterion (an egalitarian criterion discussed at length in Van Parijs (1995)) which grades '0' an achievement/affordance type of an assignment if there exists another achievement/affordance type of the same assignment that weakly dominates it, and '1' otherwise (see Van Parijs (1995) for a detailed discussion and advocacy of UD, and Basili and Vannucci (2013) for a characterization). Clearly, in order to classify an achievement/affordance assignment g^{UD} relies on *comparisons* between types of the assignment. It follows that it cannot qualify as a *sufficientarian* binary grading rule, to the extent that on the contrary the latter is supposed to rely on 'absolute' assessments on the 'sufficiency' of achievement/affordance types at any given assignment. Thus, we must introduce a list of properties of binary grading functions in order to obtain a characterization of *sufficientarian grading rules* as a proper subclass of such functions.

3.2 A characterization of sufficientarian grading rules: endogenizing thresholds

To begin with, let us observe that in order to avoid the need for clumsy clauses or qualifications it is most convenient and quite natural to consider a capability space that is large enough to include both a capability- type that is sufficient for any agent and a capability-type that, on the contrary, is not sufficient for any agent. Moreover, as already mentioned above, the capabilities we are going to focus on are meant to be observable, verifiable and *individual* (see, however, the last section of the paper for a few remarks on *public* capabilities). Accordingly, we shall focus on those binary grading functions (BGFs) $g : \mathbf{X}^n \to \{0,1\}^n$ (as defined on arbitrary capability-assignments in $\mathbf{X}^n$ where $N := \{1, ..., n\}$ is the set of relevant agents) that are *nontrivial* in the following sense: there exist $\mathbf{y}_{[n]}$,$\mathbf{z}_{[n]} \in \mathbf{X}^n$ such that $N_1(g(\mathbf{y}_{[n]})) = [n]$ and $N_0(g(\mathbf{z}_{[n]})) = [n]$ (where, for any $\mathbf{x}_{[n]} \in \mathbf{X}^n$, $N_1(g(\mathbf{x}_{[n]}))$ and $N_0(g(\mathbf{x}_{[n]}))$ denote the subsets of 1-graded and 0-graded agents at capability-assignment $\mathbf{x}_{[n]}$ according to g , respectively).

Let us now introduce the properties of BGFs to be used in order to provide our basic characterization of *sufficientarian binary grading rules.*

Definition 1 (**Isotony**) A BGF $g : \mathbf{X}^n \to \{0,1\}^n$ is *isotonic* if and only if for any $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^n$, $\mathbf{x}_{[n]} \leqslant \mathbf{x}'_{[n]}$ entails $g(\mathbf{x}_{[n]}) \leqslant g(\mathbf{x}'_N)$.

Definition 2 (**Separability**) A BGF $g : \mathbf{X}^n \to \{0,1\}^n$ is *separable* if and only if for any $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^n$ and $i \in [n]$, if $\mathbf{x}_i = \mathbf{x}'_i$ entails $(g(\mathbf{x}_{[n]}))_i = (g(\mathbf{x}'_{[n]}))_i$.

Definition 3 (**Symmetry**) A BGF $g : \mathbf{X}^n \to$ is *symmetric* if and only if for any $\mathbf{x}_{[n]} \in \mathbf{X}^n$, and permutation $\sigma : [n] \longrightarrow [n]$, $g(\mathbf{x}_{\sigma[n]}) = \sigma(g(\mathbf{x}_{[n]}))$.

Isotony of a BGF is meant to capture the fact that the components of a capability-type are uncontroversially positively correlated with well-being, and *Separability* captures the understanding that every component of a capability type denotes a *strictly individual characteristic* of an arbitrary agent. *Symmetry* amounts to an *universality* requirement concerning the assessment standards used in order to establish what is and what is not '*sufficient*': notice that (as any such standard amounts to a *system* of several distinct admissible thresholds, but Symmetry requires that system to be *unique* and apply to *everyone.*

It can be easily shown, and left to the reader to check, that a nontrivial isotonic, separable and symmetric BGF is also *onto.*

Definition 4. A BGF $g : \mathbf{X}^n \to \{0,1\}^n$ is a (binary) *sufficientarian grading rule* if and only if there exist a positive integer k and $\mathbf{x}^*_1, ..., \mathbf{x}^*_k \in \mathbf{X}$ such that $\mathbf{X}^* = \{\mathbf{x}^*_1, ..., \mathbf{x}^*_k\}$ is a *threshold system*, i.e., an *antichain* of $\mathbf{X}$ (namely $\mathbf{x}^*_j \nleqslant \mathbf{x}^*_h$ for every $j, h = 1, ..., k$ with $j \neq h$) and for every $\mathbf{x}_{[n]} \in \mathbf{X}^n$, and $i \in [n]$, $g_i(\mathbf{x}_{[n]})$ =1 if and only if $\mathbf{x}^*_j \leqslant \mathbf{x}_i$ for some $j = 1, ..., k$. We denote by $\mathcal{S}(\mathbf{X}^n)$ the class of all sufficientarian grading rules on $\mathbf{X}^n$.

Proposition 1. Let $g : \mathbf{X}^n \to \{0,1\}^n$ be a nontrivial BGF. Then, g is a sufficientarian binary grading rule if and only if it is isotonic, separable and symmetric.

Remark 1. The foregoing characterization is tight, since Isotony, Separability and Symmetry are independent properties (see the Appendix for the relevant details). Incidentally, observe that the UD binary grading function g^{UD} introduced in the previous Subsection satisfies both Isotony and Symmetry, but fails to satisfy Separability.

Notice that any sufficientarian binary grading rule $g : \mathbf{X}^n \to \{0,1\}^n$ induces by itself a *sufficientarian judgment* on -or equivalently a *sufficientarian classification* of- capability profiles in $\mathbf{X}^n$ by the following rule: a capability-profile $\mathbf{x}_{[n]} \in \mathbf{X}^n$ is *g-sufficient* if and only if $g_i(\mathbf{x}_{[n]}) = 1$ for every $i \in [n]$. Clearly enough, such a sufficientarian judgment can be also represented as a *'simple' total preorder* $\widehat{\succcurlyeq}_g$ with *precisely two* indifference classes thanks to nontriviality (indeed, ontoness) of g, namely for any $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^n$ $\mathbf{x}_{[n]} \widehat{\succcurlyeq}_g \mathbf{x}'_{[n]}$ if and only if either $\mathbf{x}_{[n]}$ is *g-sufficient* or $\mathbf{x}'_{[n]}$ is *not g-sufficient*. Arguably, such a 'simple' total preorder is *already* a sound and precise reformulation of the basic sufficientarian judgement as a ranking criterion.

However, the rest of the literature on sufficientarianism has been focussing on rankings, namely total preorders of $\mathbf{X}^n$ with no explicit limitations on the number of their admissible indifference classes (except possibly the size of the population $[n]$ of agents). But then, a further '*refined' sufficiency-count sufficientarian* total preorder $\succcurlyeq_g$ on $\mathbf{X}^n$ can be defined through g by the following most natural rule: for any $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^n$,

$$\mathbf{x}_{[n]} \succcurlyeq_g \mathbf{x}'_{[n]} \text{ if and only if } |\{i \in [n] : g_i(\mathbf{x}_{[n]}) = 1\}| \geq |\left\{i \in [n] : g_i(\mathbf{x}'_{[n]}) = 1\right\}|,$$

(or, equivalently, $\frac{|\{i\in[n]:g_i(\mathbf{x}_{[n]})=1\}|}{n} \geq \frac{|\{i\in[n]:g_i(\mathbf{x}'_{[n]})=1\}|}{n}$)[25].

Counterparts of the foregoing preorder have been indeed presented and discussed in the recent literature (notably, Alcantud, Mariotti and Veneziani (2022), and Nakada and Sakamoto (2024)). A characterization of the basic sufficientarian preorders $\widehat{\succcurlyeq}_g$, $\succcurlyeq_g$ will be provided below in the next subsection.

3.3 The sufficientarian total preorders induced by a sufficientarian grading rule through direct extension: characterizations

Let us now proceed from basic sufficientarian *ratings* to the sufficientarian *rankings* they induce on the space of capability-type assignments.

To begin with, observe that since $\{0,1\}^n$ is the boolean n-hypercube endowed with its own partial order $\geqslant$, any sufficientarian grading rule $g : \mathbf{X}^n \to \{0,1\}^n$ induces a unique $(\{0,1\}^n, \geqslant)$-monotonic and g- consistent *partial order* $\geqslant_g$ on $\mathbf{X}^n$ by the following most obvious rule: for any $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^n$, $\mathbf{x}_{[n]} \geqslant_g \mathbf{x}'_{[n]}$ if and only if $g(\mathbf{x}_{[n]}) \geqslant g(\mathbf{x}'_{[n]})$. It should also be stressed that $\geqslant_g$ is a

[25]For any set Y, $|Y|$ denotes the cardinality or size of Y.

'topped partial', namely a partial order with a (unique) *maximum* whenever g is onto. Let us now recall, for the sake of convenience, the simple and the sufficiency-count sufficientarian preorders $\widehat{\succcurlyeq}_g$ and $\succcurlyeq_g$ as introduced above in the previous section.

Definition 5 (*Simple sufficientarian preorder*)**.** Let $g : \mathbf{X}^n \to \{0,1\}^n$ be a sufficientarian grading rule. Then, the simple sufficientarian preorder $\widehat{\succcurlyeq}_g$ induced by g is defined as follows: for any $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^n$, $\mathbf{x}_{[n]} \widehat{\succcurlyeq}_g \mathbf{x}'_{[n]}$ if and only if either $g_i(\mathbf{x}_{[n]}) = 1$ for all $i \in [n]$, or $g_i(\mathbf{x}'_{[n]}) = 0$ for some $i \in [n]$.

Definition 6 (*Sufficiency-count preorder*). Let $g : \mathbf{X}^n \to \{0,1\}^n$ be a sufficientarian grading rule. Then, the sufficiency-count preorder $\succcurlyeq_g$ induced by g is defined as follows: for any $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^n$, $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^N$, $\mathbf{x}_{[n]} \succcurlyeq_g \mathbf{x}'_{[n]}$ if and only if $| \left\{ i \in [n] : g_i(\mathbf{x}_{[n]}) = 1 \right\} | \geq | \left\{ i \in [n] : g_i(\mathbf{x}'_{[n]}) = 1 \right\} |$.

It can be easily shown (and left to the reader to check) that both $\widehat{\succcurlyeq}_g$ and $\succcurlyeq_g$ are in fact extensions of the partial order $\geqslant_g$ to a *total preorder* on $\mathbf{X}^N$. Moreover, $\widehat{\succcurlyeq}_g \supseteq \succcurlyeq_g$ namely $\widehat{\succcurlyeq}_g$ is indeed a *coarser* extension of $\geqslant_g$ than $\succcurlyeq_g$, and is in fact a nontrivial extension of $\geqslant_g$ to a total preorder on $\mathbf{X}^N$, because $\widehat{\succcurlyeq}_g \neq \mathbf{X}^N \times \mathbf{X}^N$ by nontriviality of g. The following definitions and Claim make it precise in what sense it is also a most 'natural' extension of $\geqslant_g$.

Definition 7 (*The top class of a BGF*) Let $g : \mathbf{X}^n \to \{0,1\}^n$ be a BGF. Then the top class of g is $top_g(\mathbf{X}^n) := g^{-1}(\mathbf{1})$.

Notice that $\mathbf{X}^n \neq top_g(\mathbf{X}^n) \neq \emptyset$ whenever g is (as in our case) a *nontrivial* BGF.

Definition 8 (*Top-faithful sufficientarian preorders).* Let $g : \mathbf{X}^n \to \{0,1\}^n$ be a sufficientarian grading rule, and $\succcurlyeq$ a total preorder on $\mathbf{X}^n$ that is an extension of $\geqslant_g$, namely $\geqslant_g \subseteq \succcurlyeq$. Then, $\succcurlyeq$ is a top-faithful (sufficientarian) preorder induced by g if $max(\succcurlyeq) = top_g(\mathbf{X}^n)$.

Thus, in plain words, a total preorder over the set $\mathbf{X}^n$ of all possible assignments of capability-types to agents that extends the partial order induced on $\mathbf{X}^n$ by a sufficientarian grading rule is top-faithful whenever the set of its maximal elements is precisely the top class of g. The largest of such total preorders is indeed the simple sufficientarian preorder, as made precise by the following (second-order) characterization.

Claim 1 Let $g : \mathbf{X}^n \to \{0,1\}^n$ be a sufficientarian grading rule. Then, the simple sufficientarian preorder $\widehat{\succcurlyeq}_g$ is the *coarsest* top-faithful extension of $\geqslant_g$ to a total preorder $\succcurlyeq$ on $\mathbf{X}^n$.

Let us now consider the *sufficiency-count* total preorder $\succcurlyeq_g$ induced by a sufficientarian grading rule g. To begin with, observe that $\succcurlyeq_g$is also a top-faithful sufficientarian preorder induced by g. Next, in order to proceed to a characterization of $\succcurlyeq_g$ we introduce the following properties for total preorders $\succcurlyeq$ on $\mathbf{X}^n$.[26]

- ($\mathbf{X}$-*Isotony* ($\mathbf{X}$-*IS*)) A total preorder $\succcurlyeq$ on $\mathbf{X}^n$ satisfies $\mathbf{X}$-IS iff for any $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^n$, $\mathbf{x}'_{[n]} \leqslant \mathbf{x}_{[n]}$ entails $\mathbf{x}_{[n]} \succcurlyeq \mathbf{x}'_{[n]}$.

X-Isotony just express the understanding that we are focusing here on assignments of access to 'achievements' as opposed of, say, 'burdens': thus, more access to achievements cannot results in a

[26]We denote with $\sim$ and $\succ$ the symmetric and asymmetric components of $\succcurlyeq$, respectively.

lower ranking position.

- (*Anonymity (AN)*) A total preorder $\succcurlyeq$ on $\mathbf{X}^n$ is *anonymous* if and only if, for any $\mathbf{x}_{[n]} \in \mathbf{X}^n$, and any permutation $\sigma : [n] \longrightarrow [n]$, $\mathbf{x}_{[n]} \sim \mathbf{x}_{\sigma[n]}$.

Anonymity simply requires that the ranking of a capability-type assignment +does *not* depend on the identity of the agents its capability-types are assigned to.

It can be easily shown -and left to the reader to check- that Anonymity and $\mathbf{X}$-Isotony are both satisfied by the simple sufficientarian preorder $\widehat{\succcurlyeq}_g$ *and* the sufficiency-count preorder $\succcurlyeq_g$induced by an arbitrary sufficientarian binary grading rule on $\mathbf{X}^n$.

On the contrary, it can also be easily checked that the following property is satisfied by sufficiency-count preorders but *not* by simple sufficientarian preorders induced by sufficientarian grading rules.

Definition 9 (*Strict Monotonicity with respect to g (SM(g))*). Let $g : \mathbf{X}^n \to \{0,1\}^n$ be an onto binary grading function and $\succcurlyeq$ a total preorder on $\mathbf{X}^n$ which is an extension of the partial order $\geqslant_g$on $\mathbf{X}^n$, and $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^n$ and $i \in [n]$ be such that $g_l(\mathbf{x}_{[n]}) = g_l(\mathbf{x}'_{[n]})$ for any $l \in [n] \setminus \{i\}$, $g_i(\mathbf{x}_{[n]}) = 1$ and $g_i(\mathbf{x}'_{[n]}) = 0$, then $\mathbf{x}_{[n]} \succ \mathbf{x}'_{[n]}$.

Proposition 2 A total preorder $\succcurlyeq$ on $\mathbf{X}^N$ is an extension of the partial order $\geqslant_g$that satisfies AN and SM(g) if and only if $\succcurlyeq = \succcurlyeq_g$.

Remark 2. It should be noticed that the simple sufficientarian preorder $\widehat{\succcurlyeq}_g$ does satisfy AN, but fails to satisfy SM(g) whenever $n \geq 2$ (indeed, if $n = 1$ the simple sufficientarian preorder and the sufficiency-count preorder do coincide). Thus, for $n \geq 2$, the characterization of the sufficiency-count preorder $\succcurlyeq_g$ provided by Proposition 2 is indeed tight (since of course any *projection* of g induces a total preorder on $\mathbf{X}^n$ which extends partial order $\geqslant_g$ and satisfies SM(g) but violates AN whenever $n \geq 2$).

The sufficiency-count preorder $\succcurlyeq_g$ may be regarded as a special case of the cardinality total preorder of opportunity sets, and could also be characterized along the same lines of Pattanaik and Xu (1990) where three axioms are used: Indifference between no-choice situations, Independence, and Strict monotonicity. Essentially, our own characterization replaces their first two axioms with Anonymity, which is arguably much more 'natural' in the present and more specific setting.

3.4 Sufficiency-gap total preorders induced by a sufficientarian grading rule: some characterizations

Clearly enough, the sufficiency-count total preorder $\succcurlyeq_g$ on capability-type assignments induced by a sufficientarian binary grading rule g amounts to an analogue of the head-count based poverty ranking. This simple observation obviously invites the introduction of a sufficiency-gap ranking counterpart of poverty-gap rankings as an alternative refinement of the simple sufficientarian total preorder.

We introduce a sufficiency-gap ranking by jointly exploiting the structure of the capability-type space $\mathbf{X}$ *and* the characteristic threshold system $\mathcal{X}^* := \{\mathbf{x}_1^*, ..., \mathbf{x}_k^*\}$ of the sufficientarian binary grading rule g. Indeed, the capability-type space $(\mathbf{X}, \leqslant)$ with $\mathbf{X} :=_{i=1}^m \mathbf{X}_i$ is by construction a product of bounded linearly ordered sets and is therefore endowed with a natural **metric** which can be defined in several equivalent ways. In what follows, we shall introduce that metric relying on *shortest paths of the covering graph* of $\mathbf{X}$, as explained below. In order to proceed, a few new definitions are now required.

Definition 10. A *chain* of poset $(\mathbf{X}, \leqslant)$ is a set $\mathbf{Y} \subseteq \mathbf{X}$ such that for any *distinct* $\mathbf{u}, \mathbf{v} \in \mathbf{Y}$ either $\mathbf{u} \leqslant \mathbf{v}$ or $\mathbf{v} \leqslant \mathbf{u}$ holds, and its *length* $l(\mathbf{Y})$ is $|\mathbf{Y}| - 1$. A chain $\mathbf{Y}$ of $(\mathbf{X}, \leqslant)$ having x as its $\leqslant$-minimum and $\mathbf{y}$ as its $\leqslant$-maximum is *maximal* if there is no $\mathbf{z} \in \mathbf{X} \setminus \mathbf{Y}$ such that $\mathbf{x} \leqslant \mathbf{z} \leqslant \mathbf{y}$.

Definition 11. An *antichain* of poset $(\mathbf{X}, \leqslant)$ is a set $\mathbf{Y} \subseteq \mathbf{X}$ such that for any *distinct* $\mathbf{u}, \mathbf{v} \in \mathbf{Y}$ neither $\mathbf{u} \leqslant \mathbf{v}$ nor $\mathbf{v} \leqslant \mathbf{u}$ hold.

Definition 12. For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ such that $\mathbf{x} < \mathbf{y}$ (i.e. $\mathbf{x} \leqslant \mathbf{y}$ and *not* $\mathbf{y} \leqslant \mathbf{x}$) the *length* of the order-interval $[\mathbf{x}, \mathbf{y}] := \{\mathbf{z} \in \mathbf{X} : \mathbf{x} \leqslant \mathbf{z} \leqslant \mathbf{y}\}$, written $l([\mathbf{x}, \mathbf{y}])$, is the length of a (maximal) chain of *maximum* length having $\mathbf{x}$ as its $\leqslant$-minimum and $\mathbf{y}$ as its $\leqslant$-maximum. In particular, $\mathbf{x} \in \mathbf{X}$ is said to be *covered* by $\mathbf{y} \in \mathbf{X}$, written $\mathbf{x} \ll \mathbf{y}$, iff $\mathbf{x} < \mathbf{y}$ and $[\boldsymbol{x}, \boldsymbol{y}] = \{\mathbf{x}, \mathbf{y}\}$, namely $l([\mathbf{x}, \mathbf{y}]) = 1$.

Definition 13. The *covering graph* $C(\mathbf{X}) := (\mathbf{X}, E^{\ll})$ of $\mathbf{X}$ is the undirected graph having $\mathbf{X}$ as vertex-set and $E^{\ll} := \{\{\mathbf{x}, \mathbf{y}\} \subseteq \mathbf{X} : \mathbf{x} \ll \mathbf{y} \text{ or } \mathbf{y} \ll \mathbf{x}\}$ as edge-set.

Definition 14. A *path* $\pi_{\mathbf{xy}}$ of $C(\mathbf{X})$ connecting two vertices $\mathbf{x}$ and $\mathbf{y}$ is a maximal chain $\{\mathbf{z}_0, ..., \mathbf{z}_k\}$ of $\mathbf{X}$ such that $\{\mathbf{z}_0, \mathbf{z}_k\} = \{\mathbf{x}, \mathbf{y}\}$ and $\mathbf{z}_i \ll \mathbf{z}_{i+1}$, for any $i = 1, ..., k-1$, and is of *length* $l(\pi_{\mathbf{xy}}) = k$. The set of all paths of $C(\mathbf{X})$ connecting $\mathbf{x}$ and $\mathbf{y}$ is denoted by $\Pi_{\mathbf{xy}}$.

Definition 15. A *geodesic* from $\mathbf{x}$ to $\mathbf{y}$ on $C(\mathbf{X})$ is a path of *minimum length* (i.e., a *shortest path*) connecting $\mathbf{x}$ and $\mathbf{y}$.

It can be easily proved (and left to the reader to check) that the *shortest length* function $\delta : \mathbf{X} \times \mathbf{X} \to \mathbb{Z}_+$ of $C(\mathbf{X})$ as defined by the rule $\delta(\mathbf{x}, \mathbf{y}) := l(\pi_{\mathbf{xy}})$ for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ (where $\pi_{\mathbf{xy}}$ is a path of minimum length in $\Pi_{\mathbf{xy}}$) is indeed a *metric* [27].

[27] Thus, by definition, δ has *non-negative values* and satisfies the following conditions for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$:

(*i*) (*Identity Recognition*) $\delta(\mathbf{x}, \mathbf{x}) = 0$;

(*ii*) (*Identity of Indiscernibles*) $\delta(\mathbf{x}, \mathbf{y}) = 0$ only if $\mathbf{x} = \mathbf{y}$;

(*iii*) (*Symmetry*) $\delta(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{y}, \mathbf{x})$; (iv) (*Triangular Inequality*) $\delta(\mathbf{x}, \mathbf{z}) \leq \delta(\mathbf{x}, \mathbf{y}) + \delta(\mathbf{y}, \mathbf{z})$.

A sketch of the complete argument to establish validity of that statement goes as follows. By construction $\mathbf{X}$ is a (bounded) distributive lattice (i.e., it also satisfies for any $\mathbf{x}, \mathbf{y}, \mathbf{z}$,

(*distributivity*): $\mathbf{x} \vee (\mathbf{y} \wedge \mathbf{z}) = (\mathbf{x} \vee \mathbf{y}) \wedge (\mathbf{x} \vee \mathbf{z})$ or equivalently $\mathbf{x} \wedge (\mathbf{y} \vee \mathbf{z}) = (\mathbf{x} \wedge \mathbf{y}) \vee (\mathbf{x} \wedge \mathbf{z})$).

But then, $\mathbf{X}$ is also a (bounded) *modular* lattice, i.e., it satisfies for any $\mathbf{x}, \mathbf{y}, \mathbf{z}$

(*modularity) if* $\mathbf{x} \leqslant \mathbf{z}$ then $\mathbf{x} \vee (\mathbf{y} \wedge \mathbf{z}) = (\mathbf{x} \vee \mathbf{y}) \wedge \mathbf{z}$ or equivalently, if $\mathbf{z} \leqslant \mathbf{x}$ then $\mathbf{x} \wedge (\mathbf{y} \vee \mathbf{z}) = (\mathbf{x} \wedge \mathbf{y}) \vee \mathbf{z}$.

And modularity of $\mathbf{X}$, in turn, implies that (i) the length of an interval of $\mathbf{X}$ is uniquely defined (because modularity implies in particular validity of the Jordan-Dedekind chain condition for ordered sets requiring equal length of all maximal chains having the same extrema), and (ii) δ as defined above satisfies Triangular Inequality, hence it is indeed a metric (since the other three conditions are obviously satisfied).

Definition 16. Let $g : \mathbf{X}^{[n]} \to \{0,1\}^n$ be a sufficientarian grading rule, $\mathbf{x} \in X$ a capability-type and $\mathcal{X}^*(g) := (\mathbf{x}_1^*, ..., \mathbf{x}_k^*)$ the threshold system induced by g. Then, the distance $\delta_g(\mathbf{x}, \mathcal{X}^*(g))$ of $\mathbf{x}$ from $\mathcal{X}^*(g)$ is defined as

$$\delta_g(\mathbf{x}, \mathcal{X}^*(g)) := \left\{ \begin{array}{c} \min_{j\in[k]} \left\{\delta(\mathbf{x}, \mathbf{x}_j^*)\right\} \text{ if } \mathbf{x}_j^* \nleqslant \mathbf{x} \text{ for every } j \in [k], \\ 0 \text{ otherwise} \end{array} \right\}.$$

Definition 17. Let $g : \mathbf{X}^n \to \{0,1\}^n$ be a sufficientarian grading rule, $\mathbf{x}_{[n]} \in \mathbf{X}^n$ a capability-type assignment and $\mathcal{X}^*(g) := (\mathbf{x}_1^*, ..., \mathbf{x}_k^*)$ the threshold system induced by g. The *sufficiency-gap profile* of $\mathbf{x}_{[n]} \in \mathbf{X}^n$ with respect to $\mathcal{X}^*(g)$ is $(\delta_g(\mathbf{x}_i, \mathcal{X}^*(g)))_{i\in[n]} \in \mathbb{Z}_+^n$.

Definition 18. *(Gap-Antitony with respect to g (GA(g)):* Let $g : \mathbf{X}^n \to \{0,1\}^n$ be a sufficientarian grading rule, $\mathbf{x}_{[n]} \in \mathbf{X}^n$ a capability-type assignment and $\mathcal{X}^*(g) := (\mathbf{x}_1^*, ..., \mathbf{x}_k^*)$ the threshold system induced by g. Then, a preorder $\succcurlyeq$ over $\mathbf{X}^n$ is *gap-antitonic with respect to g* if, for any $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^n$, $\delta_g(\mathbf{x}_i, \mathcal{X}^*(g)) \leq \delta_g(\mathbf{x}'_i, \mathcal{X}^*(g))$ for every $i \in [n]$ implies $\mathbf{x}_{[n]} \succcurlyeq \mathbf{x}'_{[n]}$.

Definition 19. (*Sufficiency-gap total preorders induced by a sufficientarian grading rule g*) Let $g : \mathbf{X}^n \to \{0,1\}^n$ be a sufficientarian grading rule. A *sufficiency-gap preorder induced by g* is a total preorder $\succcurlyeq$ on $\mathbf{X}^n$ that satisfies Anonymity and Gap-Antitony w.r.t. g as defined above.

It should be emphasized that any sufficiency-gap preorder as defined above also satisfies by construction **X**-Isotony (precisely as sufficientarian simple preorders and sufficiency-count preorders do). On the contrary, a sufficiency-gap preorder induced by sufficientarian grading rule g need *not* be an extension of partial order $\leqslant_g$. That is so because, in general, nothing prevent existence of two capability-type assignments $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^N$ such that:

(*i*) $\delta_g(\mathbf{x}_i, \mathcal{X}^*(g)) < \delta_g(\mathbf{x}'_i, \mathcal{X}^*(g))$ and $\delta_g(\mathbf{x}'_j, \mathcal{X}^*(g)) < \delta_g(\mathbf{x}_j, \mathcal{X}^*(g))$ for some $i, j \in [n]$,

(*ii*) $|\{h \in [n] : \delta_g(\mathbf{x}_h, \mathcal{X}^*(g)) = 0\}| < |\{h \in [n] : \delta_g(\mathbf{x}'_h, \mathcal{X}^*(g)) = 0\}|$ whence $\mathbf{x}_{[n]} \succ_g \mathbf{x}'_{[n]}$, and by Symmetry of g, $\mathbf{x}'_{[n]} <_g \mathbf{x}_{[n]}$. Yet,

(*iii*) according to some plausible distance-aggregation rule, the aggregate distance of the sufficiency-gap profile of $\mathbf{x}_{[n]}$ from the threshold system induced by g on $\mathbf{X}^n$ is *also greater* than the aggregate distance of the sufficiency gap-profile of $\mathbf{x}'_{[n]}$ from that threshold system.

Of course, whether or not the foregoing conditions $(i) - (ii) - (iii)$ can be jointly satisfied ultimately depends on the possibility to specify plausible distance-aggregation rules that validate the claim under consideration. As a matter of fact, it can be easily shown that there are plenty of them. To validate that claim, it may be useful to consider a few distinguished examples of sufficiency-gap preorders, as listed below. It can be easily shown -and left to the reader to check- that each one of the following four total preorders over $\mathbf{X}^n$ satisfies both Anonymity and Gap-Antitony w.r.t. g.

- (*min-average sufficiency-gap preorder* $\succcurlyeq_{\delta_g}^{*av}$). For any $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^n$,

$$\mathbf{x}_{[n]} \succcurlyeq_{\delta_g}^{*av} \mathbf{x}'_{[n]} \text{ iff } \frac{\widetilde{\delta}_g(\boldsymbol{x}_{[n]}, \mathcal{X}^*(g))}{n} \leq \frac{\widetilde{\delta}_g(\boldsymbol{x}'_{[n]}, \mathcal{X}^*(g))}{n} \text{ iff } \widetilde{\delta}_g(\boldsymbol{x}_{[n]}, \mathcal{X}^*(g)) \leq \widetilde{\delta}_g(\boldsymbol{x}'_{[n]}, \mathcal{X}^*(g))$$

where, for any $\mathbf{z}_{[n]} \in \mathbf{X}^n$, $\widetilde{\delta}_g(\boldsymbol{z}_{[n]}, \mathcal{X}^*(g)) :=_{i\in[n]} \min \left\{\delta_g(\mathbf{z}_i, \boldsymbol{x}_j^*) : j = 1, ..., k\right\}$.

It follows that $(\mathbf{X}, \leqslant, \delta)$ is indeed a *metric* lattice. See, e.g., Barbut and Monjardet (1970) for more details.

- (*min-max sufficiency-gap preorder* $\succcurlyeq^{*\max}_{\delta_g}$). For any $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^n$,

$$\begin{array}{rcl} \mathbf{x}_{[n]} & \succcurlyeq & {}^{*\max}_{\delta_g}\mathbf{x}'_{[n]} \quad \text{if and only if} \\ max_{i\in,g[n]_i(\mathbf{x}_N)=0} \min\left\{\delta_g(\mathbf{x}_i, \boldsymbol{x}^*_j) : j=1,...,k\right\} & \leq & max_{i\in[n],g_i(\mathbf{x}'_N)=0} \min\left\{\delta_g(\mathbf{x}'_i, \boldsymbol{x}^*_j) : j=1,...,k\right\}. \end{array}$$

- (*min-leximax sufficiency-gap preorder* $\succcurlyeq^{*l\max}_{\delta_g}$). For any $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^n$, $\mathbf{x}_{[n]} \succcurlyeq^{*l\max}_{\delta_g} \mathbf{x}'_{[n]}$ if and only if there exist permutations $\sigma : [n] \longrightarrow [n]$, $\tau : [n] \longrightarrow [n]$ s.t. for any $i, j \in [n]$, $\mathbf{x}_{\sigma(i)} \geq \mathbf{x}_{\sigma(j)}$ if and only if $\sigma(i) \leq \sigma(j)$, and $\mathbf{x}'_{\sigma'(i)} \geq \mathbf{x}'_{\sigma'(j)}$ if and only if $\tau(i) \leq \tau(j)$, and there exists $h^* \in [n]$ such that $\mathbf{x}_{\sigma(i)} = \mathbf{x}'_{\tau(j)}$ for all $i, j \in [n]$ with $\sigma(i) = \tau(j) \leq h^*$, and either $h^* < n$ and $\min\left\{\delta_g(\mathbf{x}_{\tau^{-1}(h^*+1)}, \boldsymbol{x}^*_j) : j=1,...,k\right\} > \min\left\{\delta_g(\mathbf{x}_{\sigma^{-1}(h^*+1)}, \boldsymbol{x}^*_j) : j=1,...,k\right\}$, or $h^* = n$, hence $\mathbf{x}_{\sigma[n]} = \mathbf{x}'_{\tau[n]}$.

- (*min-upper-middlemost sufficiency-gap preorder* $\succcurlyeq^{*m^+}_{\delta_g}$). For any $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^n$, $\mathbf{x}_{[n]} \succcurlyeq^{*m^+}_{\delta_g} \mathbf{x}'_{[n]}$ if and only if

$m^+(\min\left\{\delta_g(\mathbf{x}_i, \boldsymbol{x}^*_j) : j=1,...,k\right\} : i \in [n], g_i(\mathbf{x}_{[n]}) = 0) \leq$
$\leq m^+(\min\left\{\delta_g(\mathbf{x}'_i, \boldsymbol{x}^*_j) : j=1,...,k\right\} : i \in [n], g_i(\mathbf{x}'_{[n]}) = 0)$
where for any $\mathbf{z} \in \mathbb{Z}^l$, $m^+(\mathbf{z})$ is the upper-middlemost value of $[\mathbf{z}] := \{z_1, ..., z_l\}$, namely $m^+(\mathbf{z}) := \max\{z_{i^*}, z_{j^*}\}$ with $i^*, j^* \in \{1, ..., l\}$ such that
$||\{z_i \in [\mathbf{z}] : z_i \leq z_{i^*}\}| - |\{z_i \in [\mathbf{z}] : z_{i^*} \leq z_i\}|| = ||\{z_i \in [\mathbf{z}] : z_i \leq z_{j^*}\}| - |\{z_i \in [\mathbf{z}] : z_{j^*} \leq z_i\}|| \in \{0, 1\}$.

Clearly, if l is odd, then $z_{i^*} = z_{j^*}$ and $m^+(\mathbf{z})$ is the *median* of $[\mathbf{z}]$: in that case $m^+(\mathbf{z})$ is the nonnegative integer z^* that minimizes the sum $\sum_{i=1}^{l} |z^* - z_i|$ and that property might be used in order to characterize $\succcurlyeq^{*m^+}_{\delta_g}$ itself as the total preorder induced by minimization of the value that minimizes the sum of modules of its differences from the individual sufficientarian-gaps: see e.g. Bandelt and Barthélemy (1984).

In what follows, we shall provide characterizations of both the min-average and the min-leximax sufficiency-gap preorders that can also be regarded as examples of sufficientarian rankings that embody generalized utilitarian and egalitarian principles, respectively.

Remark 3. As an alternative approach to identification of suitable sufficiency-gap rankings, one may start from the subclass of sufficiency-gap total preorders that also satisfy the Weak Majorization principle, as defined below. Let $g : \mathbf{X}^n \to \{0,1\}^n$ be a sufficientarian grading rule. A weak-majorization sufficientarian-gap ranking $\succcurlyeq_{\delta_g}$ induced by g is a total preorder on $\mathbf{X}^n$ which satisfies the following condition: for any $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^n$, and any pair of permutations $\sigma : N \to N$, $\sigma' : N \longrightarrow N$ such that for all $i, j \in [n]$ with $i < j$, $\delta_g(\mathbf{x}_{\sigma(j)}, \mathcal{X}^*(g)(g)) \leq \delta_g(\mathbf{x}_{\sigma(i)}, \mathcal{X}^*(g))$ and $\delta_g(\mathbf{x}'_{\sigma'(j)}, \mathcal{X}^*(g)) \leq$

$\delta_g(\mathbf{x}'_{\sigma'(i)}, \mathcal{X}^*(g))$, if for every $k \in [n]$:

$$\sum_{i\in[k],g(\sigma(i))=0} (\delta_g(\mathbf{x}_{\sigma(i)},\mathcal{X}^*(g)) \leq \sum_{i\in[k],g(\sigma(i))=0} (\delta_g(\mathbf{x}'_{\sigma(i)},\mathcal{X}^*(g))$$

then $\mathbf{x}_{[n]} \succcurlyeq \mathbf{x}'_{[n]}$.

Of course, the Weak Majorization principle by itself defines a *non-total partial preorder* on $\mathbf{X}^n$ which satisfies, by construction, both Anonymity and Gap-Antitony w.r.t. g. But then, one may obtain extensions of that partial preorder to a total preorder by using as completion criteria any one of the min-average, min-max, min-leximax, or min-upper middlemost sufficiency-gap rankings as defined above. It should also be mentioned that the Weak Majorization principle boils down to a finite family of partial sums' inequalities (indexed by $[n]$) that a partial preorder on $\mathbf{X}^n$ is required to 'respect'. In that connection, the min-average and the min-max sufficiency-gap total preorders can also be identified as those total preorders that 'respect' just *one* of the inequalities of such a family, as indexed by values $k = n$ and $k = 1$, respectively.

In order to provide characterizations of the min-average and min-leximax sufficiency-gap total preorders, we only need to observe that for every sufficientarian grading rule g on $\mathbf{X}^n$ the distances of capability-types from the threshold system $\mathcal{X}^*(g)$ of g are non-negative *integer vectors* induced by g over $\mathbf{X}^n$, and amount to a *bounded* subset $\mathcal{Z}$ of the ordered set $(\mathbb{Z}^n_+, \leqslant)$ (where $\leqslant$ denotes the natural component-wise partial order and an arbitrary element of $\mathcal{Z}$ is denoted by $\mathbf{d}$). Thus, we can reformulate the two conditions defining sufficiency-gap total preorders with no explicit reference whatsoever to sufficiency-gap profiles or sufficientarian grading rules. In particular, while the Anonymity (AN) condition stays unaltered, the family of Gap-Antitony (GA(g)) conditions are reformulated as Antitony. Then we start from the following two axioms for sufficiency-gap total preorders (we also replicate here the definition of Anonymity just for the sake of convenience):

Anonymity (AN) For any $\mathbf{d} \in \mathcal{Z}$, and every permutation π :$[n] \longrightarrow [n]$, $\mathbf{d} \sim \mathbf{d}_\pi$, where $\mathbf{d}_\pi :=$ $\left(d_{\pi(1)}, ..., d_{\pi(n)}\right)$.

Antitony (ANT) For any $\mathbf{d}$,$\mathbf{d}' \in \mathcal{Z}$ if $d_i \leq d'_i$ for all $i \in [n]$ then $\mathbf{d} \succeq \mathbf{d}'$.

A few further axioms for our characterizations are now to be introduced.

Strong Antitony (S-ANT) For any $\mathbf{d}, \mathbf{d}' \in \mathcal{Z}$ if $d_i \leq d'_i$ for all $i \in [n]$ then $\mathbf{d} \succ \mathbf{d}'$ if $\mathbf{d} \neq \mathbf{d}'$ and $\mathbf{d} \succeq \mathbf{d}'$ otherwise.

Restricted Translation Invariance (RTI) For all $\mathbf{d}, \mathbf{d}' \in \mathcal{Z}$ and $\mathbf{z} \in \mathbb{Z}^n_+$ such that both $\mathbf{d} + \mathbf{z} \in \mathcal{Z}$ and $\mathbf{d}'$+$\mathbf{z} \in \mathcal{Z}$, if $\mathbf{d} \succeq \mathbf{d}'$ then $\mathbf{d} + \mathbf{z} \succeq \mathbf{d}'$+$\mathbf{z}$.

Restricted Hammond Equity (RHE) For any $\mathbf{d}, \mathbf{d}' \in \mathcal{Z}$, if $d_i = d'_i$ for all $i \in [n] \setminus \{h, k\}$, $d_h = \max_{i\in[n]} d_i$, $d'_k = \max_{i\in[n]} d'_i$, and $d'_h \leq d_k < d_h \leq d'_k$, then $\mathbf{d} \succeq \mathbf{d}'$.[28]

[28]The label of that axiom is meant to recall that it amounts to a version of the 'Equity Axiom' first introduced by Hammond (1976, 1979) as a generalization and strengthening of the 'Weak Equity Axiom' previously suggested by Sen in the first, 1973 edition of Sen (1997), and further discussed and used in Sen (1977).

In plain words, RTI requires invariance of the preorder with respect to changes of origin of the underlying space whose points' distances are being considered. RHE requires instead that if two assignments only differ both in the points of maximal distance and in the respective value of such maximal distances, then the assignment with a smaller difference between the distances of such two points of maximum distance is either strictly better or indifferent to the other.

Proposition 3 Let $\succeq$ be a total preorder over a bounded set $\mathcal{Z} \subseteq \mathbb{Z}^n_+$, and $\succeq^{*av}$the total preorder over $\mathcal{Z}$ defined as follows: for every $\mathbf{d} := (d_1 ..., d_n), \mathbf{d}' := (d'_1, ..., d'_n) \in \mathcal{Z}$, $\mathbf{d} \succeq^{*av} \mathbf{d}'$ if and only if $\sum_{i=1}^n d_i \geq \sum_{i=1}^n d'_i$. Then, $\succeq$ satisfies AN, S-ANT and RTI if and only if $\succeq = \succeq^{*av}$.

Remark 4. The proof of the previous characterization (see the Appendix) is an adaptation and application of the 'inductive' proof technique first used in a similar setting by Sen (1977) to prove a different but related proposition, and subsequently also deployed in Hammond (1979) and d'Aspremont (1985).

Corollary 1. Let $\succeq$ be a total preorder over the bounded set $L^n_{\mathbf{X}} \subseteq \mathbb{Z}^n_+$, where $L_{\mathbf{X}} := \prod_{i=1}^m \{0, 1, ..., l_i\}$ and $L^n_{\mathbf{X}}$ is ordered according to the natural component-wise partial order of $\mathbb{Z}^n_+$. Then, $\succeq = \succcurlyeq^{*av}_{\delta_g}$ if and only if $\succeq$ satisfies AN, S-ANT and RTI.

Proof. It is easily checked that $L_{\mathbf{X}}$ is indeed the set of possible distances between points of our capability-type space $\mathbf{X}$. Then, the Corollary follows immediately from Proposition 3. □

Let us now proceed in a similar way in order to produce a characterization of the min-max aggregation rule $\succcurlyeq^{*mM}_{\delta_g}$as a special case of the characterization of min-max aggregation rules on a bounded set $\mathcal{Z} \subseteq \mathbb{Z}^n_+$.

To begin with, let us establish the validity of the following claim.

Claim 2. Let $\succeq$ be a total preorder over a bounded set $\mathcal{Z} \subseteq Z^n$ that satisfies RHE. Then $\succeq$ also satisfies AN.

The present proof of Proposition 4 below relies again (as the proof of previous Proposition 3) on the definition of a family of auxiliary total preorders $\widehat{\succeq}^m$ over $\mathcal{Z}$ indexed by $m \in [n] \setminus \{1\}$, and amounts to a 'restricted' induction argument on $[n] \setminus \{1\}$. We also need, for the sake of completeness and convenience, an explicit general definition of the min-leximax total preorder $\succeq^{*l\max}$over $\mathcal{Z}$.

Definition 20 (*min-leximax preorder* $\succeq^{*l\max}$). For any $\mathbf{d}, \mathbf{d}' \in \mathcal{Z}$, $\mathbf{d} \succcurlyeq^{*l\max} \mathbf{d}'$ if and only if there exist permutations $\sigma : [n] \longrightarrow [n]$, $\tau : [n] \longrightarrow [n]$ such that, for any $i, j \in [n]$, $\mathbf{d}_{\sigma(i)} \geq \mathbf{d}_{\sigma(j)}$ if and only if $\sigma(i) \leq \sigma(j)$, and $\mathbf{x}'_{\tau(i)} \geq \mathbf{x}'_{\tau(j)}$ if and only if $\tau(i) \leq \tau(j)$, and $h^* \in [n]$ such that $\mathbf{d}_{\sigma(i)} = \mathbf{d}'_{\tau(j)}$ for all $i, j \in [n]$ with $\sigma(i) = \tau(j) \leq h^*$, and either $[h^* < n$ and $d'_{\tau^{-1}(h^*+1)} > d_{\sigma^{-1}(h^*+1)}]$ or $h^* = n$ hence $\mathbf{d}_\sigma = \mathbf{d}'_\tau$.

Proposition 4. Let $\widehat{\succeq}$ be a total preorder over a bounded set $\mathcal{Z} \subseteq \mathbb{Z}_+^n$, and $\succeq^{*l\max}$the min-leximax total preorder over $\mathcal{Z}$. Then, $\widehat{\succeq}$ satisfies S-ANT and RHE if and only if $\widehat{\succeq} = \succeq^{*l\max}$.

Corollary 2. Let $\succeq$ be a total preorder over the bounded set $L_{\mathbf{X}}^n \subseteq \mathbb{Z}_+^n$, where $L_{\mathbf{X}} := \prod_{i=1}^m \{0,1,...,l_i\}$ and $L_{\mathbf{X}}^n$ is ordered according to the natural component-wise partial order of $\mathbb{Z}_+^n$. Then, $\succeq = \succcurlyeq_{\delta_g}^{*l\max}$ if and only if $\succeq$ satisfies S-ANT and RHE.

Proof. It is easily checked that $L_{\mathbf{X}}$ is indeed the set of possible distances between points of our capability-type space $\mathbf{X}$. Then, the Corollary follows immediately from Proposition 4. □

Let us now try and summarize the main findings of the present section. To begin with, it has been shown that a sufficientarian assessment of capability profiles as specified by its simple or sufficiency-count total preorders is a valuable and self-standing basic benchmarking criterion of distributive justice. Different considerations apply when it comes to the design or assessment of policies to improve on profiles that fail (perhaps grossly) the sufficientarian criterion itself. As mentioned above, such an assessment typically requires criteria to identify priorities concerning the agents who should be allotted more resources among those who fail to achieve the sufficiency threshold (and perhaps those agents who achieve such threshold and could afford larger transfers to the former), precisely as in the case of poverty abatement policies which typically rely on poverty-gap criteria.

Concerning such policy-related issues, we have just shown that a sufficientarian perspective can also provide a distinctive contribution to defining priority criteria to be used as a guidance or assessment of policies aimed at improving on allocations that definitely fail to meet sufficientarian criteria. But we should emphasize that, arguably, relying on sufficientarian criteria in order to assess social progress in achievement allocation is in principle consistent with the endorsement of *virtually any other criterion* when it comes to guiding or assessing remedial policies to be applied to 'non-sufficientarian' affordance/achievement allocations (e.g. utilitarian, leximin, undominated diversity, inequality reduction or more generally any suitable *prioritarian* criterion either egalitarian or not: see, among others, Atkinson and Bourguignon (1982), Moulin (1988), Sen (1997), Parfit (1997), Roemer (2004), Arneson (2006), Cohen (2011)).

4 Strategy–proof identification and selection of a sufficientarian grading rule

Let us now address the issue of designing protocols to actually implement sufficientarian grading rules (i.e. sufficientarian *BGFs).* As mentioned above, we take achievement vectors of the capabilitiy space to be observable and verifiable. However, a sufficientarian BGF g relies in fact on a unique *threshold system,* that must be somehow defined, and conversely. Thus we have to address the problem of choosing a *specific* sufficientarian BGF g, or equivalently a specific threshold system (i.e., antichain). We claim that such a threshold system can and should be determined in a consensual manner by *aggregating the opinions* on that matter of the agents involved (or perhaps of their representatives, or appointed experts). Thus, we have to supplement our model with the (true) judgements of the relevant agents concerning the appropriate 'sufficientarian' system of thresholds, by asking them to disclose such a (private) true judgement. Now, for any agent such a judgement can be regarded as her/his most preferred 'sufficientarian' g. But then, since the actual, true judgments of the relevant agents are private information of the latter since they are typically not observable and verifiable, it follows that the aggregation rules to be used should be *strategy-proof* in order to prevent strategic manipulation of outcomes (namely, the chosen threshold systems) by submission of incorrect/false judgements.

Thus, in a more formal language, the situation can be described as follows. Let $\mathcal{S}(\mathbf{X}^n)$ denote the class of all sufficientarian BGFs over $\mathbf{X}^n$ and $\mathcal{A}_{\mathbf{X}}$ the set of all antichains (or threshold systems) of $\mathbf{X}$. Our aim is to define a well-behaved protocol enabling a given set of agents/stakeholders $[n'] \subseteq [n]$ to select some specific sufficientarian BGF $g : \mathbf{X}^n \to \{0,1\}^n$, which is of course induced by one specific antichain $\mathcal{X}^* := (\mathbf{x}_1^*, ..., \mathbf{x}_k^*) \in \mathcal{A}_{\mathbf{X}}$, and $\mathcal{A}_{\mathbf{X}}$. Suppose also, for the sake of simplicity, that all agents in $[n]$ are involved, i.e., that $[n'] = [n]$. Each agent is required to propose the most appropriate sufficientarian grading rule over $\mathbf{X}^n$, or equivalently and more conveniently, the most appropriate antichain $\mathcal{X}$ of $\mathbf{X}$ (namely, $\mathcal{X} \in \mathcal{A}_{\mathbf{X}}$).

The first key point to notice is that in the present framework based upon finite capability-type space $\mathbf{X}$, the set $\mathcal{S}(\mathbf{X}^n)$ of sufficientarian grading rules over $\mathbf{X}^n$ ordered by the natural point-wise partial order $\leqslant$ [29] and the set $\mathcal{A}_{\mathbf{X}}$ of antichains of $\mathbf{X}$ endowed with weak dominance $\gtrapprox$ are essentially the same thing. This is made precise by the following definitions and lemma.

Definition 21 (*The partially ordered set* $(S(X^n), \leqslant)$ *of sufficientarian grading rules*). For any pair of sufficientarian BGFs $g, g' \in \mathcal{S}(\mathbf{X}^n)$, $g \leqslant g'$ if and only if $g(\mathbf{x}_{[n]}) \leq g'(\mathbf{x}_{[n]})$ for every $\mathbf{x}_{[n]} \in \mathbf{X}^n$. Furthermore, for every $\mathbf{x}_{[n]} \in \mathbf{X}^n$, $(g \vee g')(\mathbf{x}_{[n]}) := g(\mathbf{x}_{[n]}) \vee g'(\mathbf{x}_{[n]})$ and $(g \wedge g')(\mathbf{x}_{[n]}) := g(\mathbf{x}_{[n]}) \wedge g'(\mathbf{x}_{[n]})$.

Definition 22 (*The relational system* $(A_{\mathbf{X}}, \gtrapprox)$ *of antichains endowed with the weak dominance relation)*. For any pair of threshold systems or antichains $\mathcal{X}, \mathcal{X}' \in \mathcal{A}_{\mathbf{X}}$, $\mathcal{X} \gtrapprox \mathcal{X}'$ if and only if for every $\mathbf{x}_i \in \mathcal{X}$ there exists $\mathbf{x}'_j \in \mathcal{X}'$ such that $\mathbf{x}'_j \leqslant \mathbf{x}_i$.

We also recall here, for the sake of completeness, that a lattice $(X, \vee, \wedge)$ is *distributive* if and only if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, or equivalently $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in X$.

[29] More formally, for any pair of sufficientarian BGFs $g, g' \in \mathcal{S}(\mathbf{X}^n)$, $g \leqslant g'$ if and only if $g(\mathbf{x}_{[n]}) \leq g'(\mathbf{x}_{[n]})$ for every $\mathbf{x}_{[n]} \in \mathbf{X}^n$.

Lemma 1.The weak dominance relation $\gtrapprox$ is a partial order, hence both $(\mathcal{S}(\mathbf{X}^n), \leqslant)$ and $(\mathcal{A}_{\mathbf{X}}, \gtrapprox)$ are partially ordered sets. Moreover, $(\mathcal{S}(\mathbf{X}^n), \leqslant)$ and $(\mathcal{A}_{\mathbf{X}}, \gtrapprox)$ are in fact two *isomorphic distributive lattices.*

Such a Lemma relies heavily on a theorem due to Dilworth which establishes that the antichains (subsets of mutually incomparable elements) of a finite partially ordered set are indeed a (finite) *distributive lattice* (Dilworth (1960), see also Anderson (1987)). Then, the Lemma asserts that such a lattice is indeed, by construction, isomorphic to the lattice of sufficientarian BGFs induced by the natural point-wise partial order which is of course also a distributive lattice. But then, a natural *metric* can be defined on the *set* $\mathcal{A}_{\mathbf{X}}$ *of antichains of* $\mathbf{X}$ (or equivalently on the set of all sufficientarian BGF over $\mathbf{X}^n$ along the same lines of the geodesic-based metric on $\mathbf{X}$ defined in the previous section). To put it in simple words, such a natural metric is defined as follows: the distance between two sufficientarian BGFs $g, g' \in \mathcal{S}(\mathbf{X}^n)$ is precisely the distance between their respective threshold systems or antichains $\mathcal{X}^*(g), \mathcal{X}^*(g')$. And the distance between antichains $\mathcal{X}^*(g)$ and $\mathcal{X}^*(g')$ is the minimum distance among the distances between pairs $\mathbf{x}, \mathbf{x}' \in \mathbf{X}$ such that $\mathbf{x} \in \mathcal{X}^*(g)$ and $\mathbf{x}' \in \mathcal{X}^*(g')$. It should be noticed that such a distance on antichains of $\boldsymbol{X}$ is in fact the obvious extension to $\mathcal{A}_{\mathbf{X}}$ of the 'natural' metric d on $\mathbf{X}$ defined above [30], and makes it also possible to introduce a *metric betweenness* ternary relation B_d on $\mathcal{A}_{\mathbf{X}}$ as defined below.

Definition 23 *(Metric Betweenness over Antichains of* $\mathbf{X}$*).* The *d-metric betweenness* relation B_d is the ternary relation on $\mathcal{A}_{\mathbf{X}}$ defined by the following rule: for any $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{V} \in \mathcal{A}_{\mathbf{X}}$, $(\mathcal{X}, \mathcal{V}, \mathcal{Y}) \in B_d$ (namely, $\mathcal{V}$ is between $\mathcal{X}$ and $\mathcal{Y}$ according to the d-metric) if and only if $d(\mathcal{X}, \mathcal{Y}) = d(\mathcal{X}, \mathcal{V}) + d(\mathcal{V}, \mathcal{Y})$.[31]

It follows that, under the natural assumption that any agent regards her/his own judgement concerning the appropriate sufficiency-threshold system as the best one and considers any other judgement on that matter to be the better the *closer* it is to her/his own according to *that* 'natural' metric. Such a notion of an agent's preferences between threshold systems makes perfect sense, but requires a precise notion of *one* threshold system *being closer* than *another* to a *third* one. And the metric betweenness B_d over $\mathcal{A}_{\mathbf{X}}$ provides *exactly* that kind of notion with the resulting *ternary space* $(\mathcal{A}_{\mathbf{X}}, B_d)$[32].

[30] We are indeed denoting by d both the metrics of $\mathbf{X}$ and its 'extension' to $\mathcal{A}_{\mathbf{X}}$, which is strictly speaking a slight abuse of language, but a quite innocuous one. That is so because, by definition, any singleton $\{\mathbf{x}\}$, with $\mathbf{x} \in \mathbf{X}$, is a (degenerate) antichain of $\mathbf{X}$. It follows that one might as well start by first defining d over $\mathcal{A}_{\mathbf{X}}$ and then identifying the distance on $\mathbf{X}$ with the restriction of d to the subset of 'degenerate' singleton antichains of $\mathbf{X}$.

[31] It is well-known but worth recalling that in any bounded distributive lattice the metric betweenness is identical to both the *median betweenness* and the *interval-length betweenness* (see, e.g., Barbut and Monjardet (1970) for the relevant definitions and details).

[32] It should also be noticed that such a ternary space is in particular a *median space* since (as mentioned in the previous note) the metric betwenness B_d is also the *median betweenness* of $\mathcal{A}_{\mathbf{X}}$ (see also Nehring and Puppe (2007)).

Once this further 'intrinsic' structure of the set $\mathcal{A}_{\mathbf{X}}$ of threshold systems/antichains of $\mathbf{X}$ as a (median) ternary space is made explicit and put in place, one can immediately appreciate the 'naturality' and generality of single-peaked preferences over $(\mathcal{A}_{\mathbf{X}}, B_d)$, as defined below.

Definition 25 (*Single-Peaked Preferences over Antichains of* $\mathbf{X}$). Let $(\mathcal{A}_{\mathbf{X}}, B_d)$ be the ternary space of antichains of $\mathbf{X}$ induced by metric betweenness B_d, and $\succcurlyeq$ a preorder i.e. a reflexive and transitive binary relation over $\mathcal{A}_{\mathbf{X}}$ (we shall denote by $\succ$ and $\sim$ its asymmetric and symmetric components, respectively, by $Top(\succcurlyeq)$ the possibly empty set of its maxima, and by $||$ the set of its *incomparable* ordered pairs i.e. $x||y$ if and only if neither $x \succcurlyeq y$ nor $y \succcurlyeq x$ hold).

Then, $\succcurlyeq$ is said to be *single-peaked* in $(\mathcal{A}_{\mathbf{X}}, B_d)$ if and only if SP-(i) there is a *unique maximum* of $\succcurlyeq$ in $\mathcal{A}_{\mathbf{X}}$, its *top* antichain -denoted *top*$(\succcurlyeq)$- and SP-(ii) for all $\mathcal{X}, \mathcal{Y}, \mathcal{V} \in \mathcal{A}_{\mathbf{X}}$, if $(\mathcal{X}, \mathcal{V}, \mathcal{Y}) \in B_d$ then *not* $\mathcal{Y} \succ \mathcal{V}$.

The set of all single-peaked preference preorders in $(\mathcal{A}_{\mathbf{X}}, B_d)$ is denoted $\mathcal{D}_{B_d}$.

Observe that preferences that are *preorders with a unique maximum that are single-peaked* with respect to metric betweenness B_d are a most appropriate representation of preferences over threshold-systems/antichains of $\mathbf{X}$ (or, equivalently, sufficientarian grading rules on $\mathbf{X}^n$) such that each agent's opinion is her unique top antichain, while concerning any other antichain is regarded to be the better the closer it is to the top antichain, as dictated by B_d). Thus, for any profile of (true) judgements on the appropriate threshold-system/antichain $\mathcal{X}$ (or equivalently sufficientarian BGF g on $\mathbf{X}^n$) we end up with a 'natural' profile of single-peaked preferences on sufficientarian BGFs on $\mathbf{X}^n$).

But then, we are now in a position to consider aggregation rules for threshold-systems/antichains of $\mathbf{X}$ (or equivalently sufficientarian BGFs on and their properties, including strategy-proofness properties

Definition 26. An *aggregation rule* for $([n], \mathcal{A}_{\mathbf{X}})$ is a function $f : \mathcal{A}^n_{\mathbf{X}} \to \mathcal{A}_{\mathbf{X}}$.

Definition 27 (*Strategy-Proofness on* $\mathcal{D}^n_{B_d}$ *of an aggregation rule for* $([n], \mathcal{A}_{\mathbf{X}})$). An aggregation rule f for $([n], \mathcal{A}_{\mathbf{X}})$ is *strategy-proof* on $\mathcal{D}^n_{B_d}$ iff for all single–peaked $[n]$-profiles $(\succcurlyeq_i)_{i \in N}$ in $(\mathcal{A}_{\mathbf{X}}, B_d)$, and for all $i \in [n]$, $\mathbf{y}_i \in \mathbf{X}$, and $(\mathbf{x}_j)_{j \in [n]} \in \mathbf{X}^n$ such that $\mathbf{x}_j = top(\succcurlyeq_j)$ for each $j \in [n]$, *not* $f((\mathbf{y}_i, (\mathbf{x}_j)_{j \in ([n] \smallsetminus \{i\})})) \succ_i f((\mathbf{x}_j)_{j \in [n]})$.

Non-trivial strategy-proof aggregation rules should be -at least to some extent- *input-responsive* and *output-unbiased.* A few requirements can be deployed to present several versions and degrees of input-responsiveness and output-unbiasedness of aggregation rules, namely

Inclusiveness: an aggregation rule for $([n], \mathcal{A}_{\mathbf{X}})$ is *inclusive* if and only if for each voter $i \in [n]$ there exist $\mathbf{x}_{[n]} \in \mathbf{X}^n$ and $\mathbf{y}_i \in \mathbf{X}$ such that $f(\mathbf{x}_{[n] \smallsetminus \{i\}}, \mathbf{y}_i) \neq f(\mathbf{x})$.

Anonymity: an aggregation rule f for $([n], \mathcal{A}_{\mathbf{X}})$ is *anonymous* if for each $\mathbf{x}_{[n]} \in \mathbf{X}^N$ and each permutation $\sigma : [n] \longrightarrow [n]$, $f(\mathbf{x}_{[n]}) = f(\mathbf{x}_{\sigma[n]})$ (where $\mathbf{x}_{\sigma[n]} = (\mathbf{x}_{\sigma(1)}, ..., \mathbf{x}_{\sigma(n)})$).

Idempotence: an aggregation rule f for $([n], \mathcal{A}_{\mathbf{X}})$ is *idempotent* (or *unanimity-respecting)* if $f(\mathbf{x}, ..., \mathbf{x}) = \mathbf{x}$ for each $\mathbf{x} \in \mathbf{X}$.

Sovereignty: an aggregation rule f for $([n], \mathcal{A}_{\mathbf{X}})$ is *sovereign* if for each $\mathbf{y} \in \mathbf{X}$ there exists $\mathbf{x}_{[n]} \in \mathbf{X}^n$ such that $f(\mathbf{x}_{[n]}) = \mathbf{y}$ i.e. f is an *onto* function.

Neutrality: an aggregation rule f for $([n], \mathcal{A}_{\mathbf{X}})$ is *neutral* if for each $\boldsymbol{x}_{[n]} \in \mathbf{X}^n$ and each permutation π: $\mathbf{X} \longrightarrow \mathbf{X}$, $f(\pi(\mathbf{x}_{[n]})) = \pi(f(\mathbf{x}_{[n]}))$ (where $\pi(\mathbf{x}_{[n]}) = (\pi(\mathbf{x}_1), ..., \pi(\mathbf{x}_n))$).

Notice that both *Idempotence* and *Neutrality* imply *Sovereignty* (but not conversely), while *Anonymity* and *Sovereignty* jointly imply *Inclusiveness* (but not conversely). However, it is easily checked that *if Strategy-Proofness holds,* Sovereignty and Idempotence are in fact equivalent.

Now, we are looking for some aggregation rule $G : (\mathcal{S}(\mathbf{X}^n)\ )^n \longrightarrow \mathcal{S}(\mathbf{X}^n)$ that for any profile of proposed sufficientarian BGFs returns a single sufficientarian BGF and respects three key properties of any *reliable* 'democratic' aggregation protocol: *anonymity* (it is the list of proposals that counts not the identity of the proponent of each proposal), *idempotence* or respect for unanimity (if everyone makes the same proposal, that proposal must the chosen one), and *strategy-proofness* (nobody should be ever in the position to obtain the choice of a proposal that is better, i.e., closer to his/her truly preferred proposal by making a proposal *other than* the latter) with respect to domain D of single-peaked preferences over $\mathcal{A}_{\mathbf{X}}$ as defined above.

Let us now proceed to consider the address the task of designing and implementing an aggregation rule $G : (\mathcal{S}(\mathbf{X}^n)\ )^n \longrightarrow \mathcal{S}(\mathbf{X}^n)$ that satisfy *anonymity, idempotence and strategy-proofness* by means of a protocol that relies on the isomorphic bijection $\varphi : \mathcal{S}(\mathbf{X}^n) \longrightarrow \mathcal{A}_{\mathbf{X}}$ between sufficientarian BGFs over $\mathbf{X}^n$ and antichains of $\mathbf{X}$.

(i) To begin with, every agent $i \in [n]$ submits a sufficientarian BGF $g^i \in \mathcal{S}(\mathbf{X}^n)$ so that a profile $\mathbf{g}_{[n]} = (g^1, ..., g^n) \in (\mathcal{S}(\mathbf{X}^n)\ )^n$.

(ii) For any g^i we consider $\varphi(g^i)$ the antichain (or order filter) of $\mathbf{X}$ also denoted $\mathcal{X}_i(g^i) \in \mathcal{A}_{\mathbf{X}}$, and obtain a profile $\varphi\left(\mathbf{g}_{[n]}\right) := (\varphi(g^1), ..., \varphi(g^n))$ as an input to an aggregation rule $F : (\mathcal{A}_{\mathbf{X}})^n \longrightarrow \mathcal{A}_{\mathbf{X}}$ that returns an antichain $\mathcal{X}(\mathbf{g}_{[n]}) := F((\mathcal{X}_1(g^1), ..., \mathcal{X}_n(g^n))$ as its output.

(iii) Finally, we define $G(\mathbf{g}_{[n]}) := \varphi^{-1}(\mathbf{X}(\mathbf{g}_{[n]}))$. Moreover, we say that G is truthfully implementable or strategy-proof on a certain domain or sufficientarian BGFs if F is strategy-proof on the corresponding domain $\mathcal{D}^n_{B_d}$of (single-peaked) preorders over $\mathcal{A}_{\mathbf{X}}$.

Clearly enough, each one of the three properties Anonymity, Idempotence, and Strategy-Proofness holds for G if and only if it also holds for F.

Here, we can also take avail of the following Lemma which summarizes some previous results on inclusive, anonymous and idempotent aggregation rules over bounded distributive lattices that are also strategy-proof on suitably defined single-peaked domains (such a Lemma is directly implied by

Theorem 1 of Savaglio and Vannucci (2019), but see also Monjardet (1990), Nehring and Puppe (2007), and Vannucci (2019) for strictly related results).

Lemma 2. Let $(X, \leqslant)$ be a bounded distributive lattice and B_d its metric betweenness relation. Then there is a class of anonymous and idempotent aggregation rules for $([n], X)$ that are also single-peaked in (X, B_d). Moreover, such a class includes the simple majority rule if n is odd.

The following proposition shows the existence of aggregation rules G for sufficientarian BGF that are anonymous, idempotent and strategy-proof by showing the existence a class of aggregation rules $F : (\mathcal{A}_{\mathbf{X}})^n \longrightarrow \mathcal{A}_{\mathbf{X}}$ that are indeed anonymous, idempotent and strategy-proof on the 'natural' and large domain of single-peaked preorders on $\mathcal{A}_{\mathbf{X}}$ defined above, and also enable implementation of the corresponding G.

Proposition 5. Suppose n is an odd number. Then, the simple majority aggregation rule $G : (\mathcal{S}(\mathbf{X}^n\ ))^n \longrightarrow \mathcal{S}(\mathbf{X}^n)$ for sufficientarian BGFs on capability profiles in $\mathbf{X}^n$ is strategy-proof, namely it is implementable by a protocol $\varphi_{[n]}^{-1} \circ F \circ \varphi^{-1}$ (where $F : (\mathcal{A}_{\mathbf{X}})^n \longrightarrow \mathcal{A}_{\mathbf{X}}$ is an aggregation rule which is *strategy-proof* on the domain $\mathcal{D}_{B_d}^n$ of preference profiles over $\mathcal{A}_{\mathbf{X}}$ that are single-peaked in $(\mathcal{A}_{\mathbf{X}}, B_d)$, and φ is an isomorphism of $\mathcal{S}(\mathbf{X}^n)$ and $\mathcal{A}_{\mathbf{X}}$).

Remark 6. The foregoing protocol can be further refined by adjoining to it a subprotocol consisting of a further strategy-proof aggregation rule to endogenously select a committee of representative agents (or expert agents) out of $[n]$ by the agents in $[n]$ themselves (e.g. by repeatedly sampling $[n]$ without replacement, via *uniform random dictatorship* as implemented through a suitable modular arithmetic component). Notice that such a subprotocol might also be used in order to select *'endogenously'* on every single occasion a *president* endowed with a double vote when n is even, to the effect of making the practical import of the *oddness* requirement for n of Proposition 5 a quite minor one. Furthermore, the same protocol can be easily extended to cover any case under which more than one threshold is to be elicited (say, the sufficiency threshold system and a typically higher 'limitarian' one to establish preferential eligibility as targets of redistributive taxation, and/or a typically lower poverty threshold). To address such a double-threshold elicitation problem just consider the product antichain space $\mathcal{A}_{\mathbf{X}} \times \mathcal{A}_{\mathbf{X}}$ (which is also finite distributive lattice by construction) as the new individual strategy space. Each agent is required to select an ordered pair of antichains in $\mathcal{A}_{\mathbf{X}}$: the two required threshold systems may be obtained by aggregating, respectively, the meets and joins of each one of the submitted individual pairs. A similar approach can be taken to address the task of selecting any *finite* number of threshold systems.

5 Concluding remarks

In this paper, under the general label of 'sufficientarian grading rules', basic sufficientarian rating rules have been introduced and characterized, relying on binary grading functions (BGFs) as defined on a finite capability-type space which consists in a finite product of finite linear orders. And, remarkably, the characterization of basic sufficientarian rating rules so provided only requires a version of three independent conditions (i.e., Symmetry, Separability, and Isotony) that are largely used in the extant literature, eschewing any reference to thresholds. Moreover, such a characterization highlights a one-to-one correspondence between basic sufficientarian rules and *threshold systems* given by sets of mutually incomparable (or antichains of) threshold-points of the capability-type space. Sufficientarian *ranking* rules induced by basic sufficientarian rating rules are also defined and characterized, including *sufficiency-count* ranking rules and a pair of *sufficiency-gap* ranking rules.

Furthermore, and last but not least, the largely neglected but crucial issue concerning the identification of practical ways to select one specific threshold system has been squarely addressed from a mechanism-design perspective. In particular, it has been shown that nicely democratic and strategy-proof opinion aggregation rules on the set of threshold-systems do exist, and can be easily deployed to design the required protocols for threshold-system selection. When it comes to implementation of sufficientarian rules, It should be fully appreciated the expediency of such protocols. Of course, selection of any specific sufficientarian rating or ranking rule as required for any practical application amounts to choosing precisely one threshold system. However, as previously noticed and discussed, adoption of a sufficientarian stance may well suggest consideration and possibly specification of *further* threshold systems (perhaps a 'higher' and/or a 'lower' one in addition to the sufficiency-threshold system itself). But then, the very same protocol may well be used *repeatedly* to select *all* the required threshold systems. Moreover, it should be noticed that, as previously mentioned, BGFs and the rating rules they induce may be used in other contexts including *poverty* analysis, *exploitation* assessment, and *expertise* evaluation, where identification of thresholds and possibly caps could have a pivotal role. In all of those cases, the class of protocols introduced in the present work might provide a useful blueprint of sorts.

Finally, we should also like to mention a couple of possible significant extensions of sufficientarian grading rules as defined in this paper.

To begin with, let us get back to the issue of *burden* distribution, which is perceived as a major challenge to sufficientarian principles by some authors, including both advocates and critics of sufficientarianism (see, e.g., Nielsen (2019) and Knight (2022)). We surmise that such an issue can be successfully addressed relying on a relatively minor extension of the sufficientarian grading rules as defined in this paper. Namely, it suffices to enlarge the capability-type space adjoining a finite set of *costly* affordances/qualifications to denote burdens. Indeed, such affordances/qualifications can also be represented by a finite number of linearly ordered ranks denoted by nonnegative numbers in decreasing order (with 0 as the top rank, the most costly one). Accordingly, new extended sufficientarian grading rules can be defined on such enlarged capability-type space. But then, the rest of the analysis provided in the previous sections can be replicated on such an extended space.

There is also a further extension of the BGF-based approach to sufficientarianism we should mention. In the present paper individual affordances/achievements related to *public goods* have been explicitly ignored. However our model can be easily extended to cover such public-good related capabilities, including *global* public goods of which basic scientific research activities and outputs are a prominent example. In a well-known and most influential paper, Merton described the rules underlying proper and successful scientific research activities in terms of four principles summarized by the acronym CUDOS (namely, 'communism', 'universalism', 'disinterestedness', 'organized skepticism': see Merton (1942)). Arguably, an augmented version of sufficientarianism as discussed above with a specific public-good related component of the capability-type space (possibly enriched with a 'burden'-subspace as discussed above) might be fruitfully used to contribute a precise version of distribution rules emboding at least some of the four Merton's principles, including at least the first two of his well-known and highly regarded list. However, further details concerning those possible extensions of the model presented in this paper are best left as a possible topic for future research.

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6 Appendix

Proof of Proposition 1

($\Longrightarrow$) Suppose g is sufficientarian, and consider any $\mathbf{x}_{[n]}$, $\mathbf{x}'_{[n]} \in \mathbf{X}^n$ with $\mathbf{x}_i = \mathbf{x}'_i$. By definition, there exists an antichain $(\mathbf{x}_1^*, ..., \mathbf{x}_k^*)$ of $\mathbf{X}$ such that for all $i \in [n]$, there is an $\mathbf{x}_h^*$, $h = 1, ..., k$, with $g_i(\mathbf{x}_{[n]}) =1$ if $\mathbf{x}_h^* \leqslant \mathbf{x}_i$ iff $\mathbf{x}_h^* \leqslant \mathbf{x}'_i$ iff $g_i(\mathbf{x}'_{[n]}) =1$. Hence, g is of course separable. Moreover, consider any $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^n$ with $\mathbf{x}_{[n]} \leqslant \mathbf{x}'_{[n]}$, and any $i \in N$. If $g_i(\mathbf{x}_{[n]}) =0$ there is nothing to prove. So, suppose that $g_i(\mathbf{x}_{[n]}) =1$. Hence, by definition, there is an $\mathbf{x}_j^* \leqslant \mathbf{x}_i \leqslant \mathbf{x}'_i$, thus, $g_i(\mathbf{x}'_{[n]}) =1$. It follows that g is also isotonic. Finally, consider any $\mathbf{x} \in \mathbf{X}^{[n]}$, any permutation $\sigma : [n] \longrightarrow [n]$ and $i, j \in [n]$ with $j := \sigma(i)$. By definition, $g_i(\mathbf{x}_{[n]}) =1$ if and only if $g_j(\mathbf{x}_{[n]}) = g_{\sigma(i)}(\mathbf{x}_{[n]}) = g_i(\mathbf{x}_{\sigma[n]}) = 1$. Thus, g is indeed symmetric.

($\Longleftarrow$) Suppose g is isotonic, separable and symmetric, and consider any $\mathbf{x}_{[n]} \in \mathbf{X}^n$, $i \in [n]$ and $g_i^{-1}(1) := \{\mathbf{x}_{[n]} \in \mathbf{X}^n : g_i(\mathbf{x}_{[n]}) = 1\}$.

Now, for any $i \in [n]$, take $\mathbf{X}(i, 1, g) := \{\mathbf{x} \in \mathbf{X} : \mathbf{x} = \mathbf{x}_i \text{ for some } \mathbf{x}_{[n]} \in g_i^{-1}(1)\}$, and posit $\mathbf{X}^u(i, 1, g) := \{\mathbf{x} \in \mathbf{X} : \mathbf{x}_i \leqslant \mathbf{x} \text{ for some } \mathbf{x} \in \mathbf{X}(i, 1, g)\}$, and observe that by *isotony* and *separability* $\mathbf{X}^u(i, 1, g) \subseteq \mathbf{X}(i, 1, g)$, whence, in fact, $\mathbf{X}(i, 1, g) = \mathbf{X}^u(i, 1, g)$. It follows that any $\mathbf{X}(i, 1, g)$ is actually an *order filter* of the partially ordered set $\mathbf{X}$ (which is a finite product of finite linearly ordered sets, hence, in particular, a finite distributive lattice). Notice however that any such order filter $\mathbf{X}^u(i, 1, g)$ is uniquely determined by its *basis* namely the set $\mathbf{X}^u_{\min}(i, 1, g)$ of its *minimal* elements, and as it is easily checked it must be the case for every $\mathbf{x}, \mathbf{x}' \in \mathbf{X}^u_{\min}(i, 1, g)$ if $\mathbf{x} \neq \mathbf{x}'$ then $\mathbf{x} \nleqslant \mathbf{x}'$ namely $\mathbf{X}^u_{\min}(i, 1, g)$ is by construction a (finite) *antichain* $(\mathbf{x}_{i1}, ..., \mathbf{x}_{ik})$ of $\mathbf{X}$. Moreover, symmetry implies that $\mathbf{X}^u_{\min}(i, 1, g) = \mathbf{X}^u_{\min}(j, 1, g) = \mathcal{X}^* := (\mathbf{x}_1^*, ..., \mathbf{x}_k^*)$ for any $i, j \in [n]$ (otherwise there is no guarantee that for every i, j and $\mathbf{x}_{[n]}$ both $g_i(\mathbf{x}_{[n]}) =1$ and $g_i(\mathbf{x}_{\sigma[n]}) = 1$ hold). But then, it follows that for any $\mathbf{x}_{[n]} \in \mathbf{X}^n$ and any $i \in [n]$, $g_i(\mathbf{x}_{[n]}) = 1$ if and only if there exists $\mathbf{x}_h^* \in \{\mathbf{x}_1^*, ..., \mathbf{x}_k^*\}$ such that $\mathbf{x}_h^* \leqslant \mathbf{x}_i$, i.e. g is indeed a *sufficientarian BGF* as required.

The foregoing characterization of sufficientarian BGFs is tight. To check validity of that statement consider the following three examples:

(*i*) $g' : \mathbf{X}^n \to \{0, 1\}^n$ is such that there exist a positive integer k and $\mathbf{x}_1^*, ..., \mathbf{x}_k^* \in \mathbf{X}$ with $\mathcal{X}^* = (\mathbf{x}_1^*, ..., \mathbf{x}_k^*)$ being an *antichain* of $\mathbf{X}$ (namely $\mathbf{x}_j^* \nleqslant \mathbf{x}_h^*$ for every $j, h = 1, ..., k$ with $j \neq h$) and for every $\mathbf{x}_N \in \mathbf{X}^N$, $g'_i(\mathbf{x}_N) =1$ if and only if $\mathbf{x}_i \leqslant \mathbf{x}_h^*$ for some $h = 1, ..., k|$. It can be checked that g' is separable and symmetric but not isotonic (indeed, it is antitonic);

(*ii*) $g'' : \mathbf{X}^n \to \{0, 1\}^n$ is such that there exist a positive integer k and $\mathbf{x}_1^*, ..., \mathbf{x}_k^* \in \mathbf{X}$ with $\mathcal{X}^* = (\mathbf{x}_1^*, ..., \mathbf{x}_k^*)$ being an *antichain* of $\mathbf{X}$ and for every $\mathbf{x}_{[n]} \in \mathbf{X}^n$, $g''_i(\mathbf{x}_{[n]}) =1$ if and only if $\mathbf{x}_i \leqslant \mathbf{x}_h^*$ and $\mathbf{x}_j \leqslant \mathbf{x}_h^*$ for some $h = 1, ..., k$, and $j \in [n] \setminus \{i\}$. Clearly, g'' is isotonic and symmetric but not separable.

(*iii*) $g''' : \mathbf{X}^n \to \{0, 1\}^n$ is such that for every i there exist a positive integer k_i and $\mathbf{x}_1^*, ..., \mathbf{x}_{k_i}^* \in \mathbf{X}$ with $(\mathbf{x}_1^*, ..., \mathbf{x}_{k_i}^*)$ being an *antichain* of $\mathbf{X}$ and $k_i \neq k_j$ for some $i, j \in [n]$, and for every $\mathbf{x}_{[n]} \in \mathbf{X}^n$,

$g'''_i(\mathbf{x}_{[n]})$ =1 if and only if $\mathbf{x}_i \leqslant \mathbf{x}^*_h$ for some $h = 1, ..., k_i$. It can be checked that g''' is separable and isotonic but not symmetric. □

Proof of Claim 1

Let $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^n$ and $\mathbf{x}_{[n]} \geqslant_g \mathbf{x}'_{[n]}$, i.e. by definition $g(\mathbf{x}_{[n]}) \geqslant g(\mathbf{x}'_{[n]})$. Two cases are to be distinguished : (i) $g_i(\mathbf{x}_{[n]}) = 1$ for all $i \in [n]$ and (ii) $g_i(\mathbf{x}_{[n]}) = 0$ for some $i \in [n]$. In the first case, $\mathbf{x}_{[n]} \widehat{\succcurlyeq}_g \mathbf{x}'_{[n]}$ by definition. In the second case $0 = g_i(\mathbf{x}_{[n]}) \geq g_i(\mathbf{x}'_{[n]}) \geq 0$ whence $g_i(\mathbf{x}'_{[n]}) = 0$ which in turn implies $\mathbf{x}_{[n]} \widehat{\succcurlyeq}_g \mathbf{x}'_{[n]}$, by definition. It follows that $\widehat{\succcurlyeq}_g$ is indeed an *extension* of $\geqslant_g$. It is also easily checked that $\widehat{\succcurlyeq}_g$ is a total preorder. That is so because both *reflexivity* and *connectedness* of $\widehat{\succcurlyeq}_g$ follow immediately by its definition. Moreover, $\widehat{\succcurlyeq}_g$ is also transitive: indeed, suppose that $\mathbf{x}_{[n]} \widehat{\succcurlyeq}_g \mathbf{x}'_{[n]}$ and $\mathbf{x}'_{[n]} \widehat{\succcurlyeq}_g \mathbf{x}''_{[n]}$. Then, again, cases (i) and (ii) are to be distinguished concerning $\mathbf{x}_{[n]}$. If (i) holds, then $\mathbf{x}_{[n]} \widehat{\succcurlyeq}_g \mathbf{x}''_{[n]}$ by definition. If on the contrary (ii) holds, $\mathbf{x}_{[n]} \widehat{\succcurlyeq}_g \mathbf{x}'_{[n]}$ implies that $g_i(\mathbf{x}'_{[n]}) = 0$ for some $i \in [n]$. But then, $\mathbf{x}'_{[n]} \widehat{\succcurlyeq}_g \mathbf{x}''_{[n]}$ implies in turn that $g_i(\mathbf{x}''_{[n]}) = 0$ for some $i \in [n]$. It follows again that $\mathbf{x}_{[n]} \widehat{\succcurlyeq}_g \mathbf{x}''_{[n]}$ by definition, *transitivity* of $\widehat{\succcurlyeq}_g$ is thus confirmed, and $\widehat{\succcurlyeq}_g$ is a well-defined total preorder.

To check that $\widehat{\succcurlyeq}_g$ is also *top-faithful*, observe that by definition if $\mathbf{x}_{[n]} \widehat{\succcurlyeq}_g \mathbf{x}'_{[n]}$ for every $\mathbf{x}'_{[n]} \in \mathbf{X}^n$ then, since g is onto, it must be the case that $g_i(\mathbf{x}_{[n]}) = 1$ for all $i \in [n]$, which in turn implies that $\mathbf{x}_{[n]} \in top_g(\mathbf{X}^{[n]})$. Conversely, if $\mathbf{x}_{[n]} \in top_g(\mathbf{X}^n)$ then by definition $g(\mathbf{x}_{[n]}) = \mathbf{1}$, which in turn implies that $\mathbf{x}_{[n]} \widehat{\succcurlyeq}_g \mathbf{x}'_{[n]}$ for every $\mathbf{x}'_{[n]} \in \mathbf{X}^n$, by definition of $\widehat{\succcurlyeq}_g$ itself.

Finally, suppose that $\succcurlyeq$ is a total preorder on $\mathbf{X}^n$ which is also a top-faithful extension of $\geqslant_g$, and consider any pair $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^n$ such that $\mathbf{x}_{[n]} \succcurlyeq \mathbf{x}'_{[n]}$. Two cases are to be distinguished: (i) $\left\{\mathbf{x}_{[n]}, \mathbf{x}'_{[n]}\right\} \cap top_g(\mathbf{X}^n) \neq \emptyset$. In this case, since by assumption $top_g(\mathbf{X}^n) = \max(\succcurlyeq) = \max(\widehat{\succcurlyeq}_g)$ it must be the case that $\mathbf{x}_{[n]} \widehat{\succcurlyeq}_g \mathbf{x}'_{[n]}$ as well, by construction; (ii) $\left\{\mathbf{x}_{[n]}, \mathbf{x}'_{[n]}\right\} \cap top_g(\mathbf{X}^n) = \emptyset$. But then, both $\mathbf{x}_{[n]} \widehat{\succcurlyeq}_g \mathbf{x}'_{[n]}$ and $\mathbf{x}'_{[n]} \widehat{\succcurlyeq}_g \mathbf{x}_{[n]}$ do hold, by construction of $\widehat{\succcurlyeq}_g$. It follows that $\succcurlyeq \subseteq \widehat{\succcurlyeq}_g$ as required, and the proof of our Claim is now complete. □

Proof of Proposition 2

To begin with, notice that $\succcurlyeq_g$ is by construction both a total preorder on X^n and an extension of $\geqslant_g$. That $\succcurlyeq_g$ satisfies AN and SM is also clearly the case. Indeed, for any $\mathbf{x}_{[n]} \in \mathbf{X}^n$ and any permutation $\sigma : [n] \longrightarrow [n]$, $|\left\{i \in [n] : g_i(\mathbf{x}_{[n]}) = 1\right\}| = |\left\{i \in [n] : g_i(\mathbf{x}_{\sigma[n]}) = 1\right\}|$ by construction. Thus $\mathbf{x}_{[n]} \sim_g \mathbf{x}_{\sigma[n]}$ (where $\sim_g$ is of course the symmetric component of $\succcurlyeq_g$) and AN holds. Moreover, let $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^n$ and $i \in [n]$ be such that $g_l(\mathbf{x}_{[n]}) = g_l(\mathbf{x}'_{[n]})$ for any $l \in [n] \setminus \{i\}$, $g_i(\mathbf{x}_{[n]}) = 1$ and $g_i(\mathbf{x}'_{[n]}) = 0$. Then, by definition $|\left\{i \in [n] : g_i(\mathbf{x}_{[n]}) = 1\right\}| = |\left\{i \in [n] : g_i(\mathbf{x}'_{[n]}) = 1\right\}| + 1 > |\left\{i \in [n] : g_i(\mathbf{x}'_{[n]}) = 1\right\}|$. Therefore, $\mathbf{x}_{[n]} \succ_g \mathbf{x}'_{[n]}$ and SM also holds.

Conversely, let $\succcurlyeq$ be a total preorder on $\mathbf{X}^n$ that is an extension of the partial order $\geqslant_g$ and satisfies both AN and SM, and consider any $\mathbf{x}_{[n]}, \mathbf{x}'_{[n]} \in \mathbf{X}^n$. Next, suppose w.l.o.g. that $|\left\{i \in [n] : g_i(\mathbf{x}_{[n]}) = 1\right\}| \geq |\left\{i \in [n] : g_i(\mathbf{x}'_{[n]}) = 1\right\}|$.

We may distinguish two cases:

(i) $|\{i\in[n]:g_i(\mathbf{x}_{[n]})=1\}| = |\left\{i\in[n]:g_i(\mathbf{x}'_{[n]})=1\right\}|$. If that is the case, then there exists a permutation $\sigma:[n]\to[n]$ such that, for any $i\in[n]$, $g_i(\mathbf{x}_{[n]})=1$ if and only if $g_{\sigma(i)}(\mathbf{x}'_{\sigma[n]})=1$. Now, $\mathbf{x}'_{[n]}\sim\mathbf{x}'_{\sigma[n]}$ by Anonymity of $\succcurlyeq$, and $g(\mathbf{x}_{[n]})=g(\mathbf{x}'_{\sigma[n]})$. Therefore, $\mathbf{x}_{[n]}\sim\mathbf{x}'_{\sigma[n]}$ since $\succcurlyeq$ is an extension of $\geqslant_g$. It follows that $\mathbf{x}_{[n]}\sim\mathbf{x}'_{[n]}$ by transitivity of $\succcurlyeq$.

(ii) $|\{i\in[n]:g_i(\mathbf{x}_{[n]})=1\}| > |\left\{i\in[n]:g_i(\mathbf{x}'_{[n]})=1\right\}|$,

with $|\{i\in[n]:g_i(\mathbf{x}_{[n]})=1\}|=k$, and $|\left\{i\in[n]:g_i(\mathbf{x}'_{[n]})=1\right\}|=h$. If that is the case, there exist a permutation

$\sigma^*:[n]\to[n]$ and $i_1^*,...,i_{k--h}^*\in\{i\in[n]:g_i(\mathbf{x}_{[n]})=1\}\setminus\left\{i\in[n]:g_i(\mathbf{x}'_{[n]})=1\right\}$ such that, for any $i\in[n]$,

$g_i(\mathbf{x}_{\sigma^*[n]})=1$ iff $i\in\{i_1^*,...,i_{k--h}^*\}\cup\left\{i\in[n]:g_i(\mathbf{x}'_{[n]})=1\right\}$.

Therefore, by construction of $\mathbf{x}_{\sigma^*[n]}$ and Anonymity of $\succcurlyeq$, it follows that $\mathbf{x}_{[n]}\sim\mathbf{x}_{\sigma^*[n]}$. Next, consider a sequence $\mathbf{x}^j_{\sigma^*[n]}\in\mathbf{X}^n$, $j=0,1,...,k-h$ defined as follows: $\mathbf{x}^j_{\sigma^*[n]}=(\mathbf{x}_{\sigma^*[n]|[n]^{*j}},\mathbf{x}'_{[n]|[n]\setminus[n]^{*j}})$ where $[n]^{*j}:=\left\{i\in[n]:g_i(\mathbf{x}'_{[n]})=1\right\}\cup\{i_1^*,...,i_j^*\}$ for $j=1,...,k-h$, while $\mathbf{x}^j_{\sigma^*[n]}=\mathbf{x}'_{[n]}$ for $j=0$. Moreover, observe that, by construction of that sequence and Strict Monotonicity w.r.t. g of $\succcurlyeq$, $\mathbf{x}^{j+1}_{\sigma^*[n]}\succ\mathbf{x}^j_{\sigma^*[n]}$, for any $j=0,1,...,k-(h+1)$. Hence, in particular, $\mathbf{x}^1_{\sigma^*[n]}\succ\mathbf{x}'_{[n]}$, while $\mathbf{x}^{k-h}_{\sigma^*[n]}=\mathbf{x}_{\sigma^*[n]}\sim\mathbf{x}_{[n]}$. As a consequence, $\mathbf{x}_{[n]}\succ\mathbf{x}'_{[n]}$ holds, by transitivity of $\succcurlyeq$. It follows that $|\{i\in[n]:g_i(\mathbf{x}_{[n]})=1\}|\geq|\left\{i\in[n]:g_i(\mathbf{x}'_{[n]})=1\right\}|$ implies $\mathbf{x}_{[n]}\succcurlyeq\mathbf{x}'_{[n]}$. Hence $\succcurlyeq_g\subseteq\succcurlyeq$. Now, suppose that for some $\mathbf{x}_{[n]},\mathbf{x}'_{[n]}\in\mathbf{X}^n$, both $\mathbf{x}_{[n]}\succcurlyeq\mathbf{x}'_{[n]}$ and *not* $\mathbf{x}_{[n]}\succcurlyeq_g\mathbf{x}'_{[n]}$ hold. Since $\succcurlyeq_g$ is a total preorder, it must be the case that $\mathbf{x}'_{[n]}\succ_g\mathbf{x}_{[n]}$ and thus $\mathbf{x}'_{[n]}\succcurlyeq\mathbf{x}_{[n]}$. Therefore, by definition $|\left\{i\in[n]:g_i(\mathbf{x}'_{[n]})=1\right\}|>|\{i\in[n]:g_i(\mathbf{x}_{[n]})=1\}|$.But then, there exist $\mathbf{x}''_{[n]},\mathbf{x}'''_{[n]}\in\mathbf{X}^n$ such that $\mathbf{x}''_{[n]}\sim_g\mathbf{x}_{[n]}$, $\mathbf{x}'''_{[n]}\sim_g\mathbf{x}'_{[n]}$ and $\mathbf{x}'''_{[n]}>_g\mathbf{x}''_{[n]}$. Thus, $\mathbf{x}'''_{[n]}\succ\mathbf{x}''_{[n]}$ as well since $\succcurlyeq$ is also an extension of $\geqslant_g$. On the other hand $\succcurlyeq_g\subseteq\succcurlyeq$, $\mathbf{x}''_{[n]}\sim_g\mathbf{x}_{[n]}$ and $\mathbf{x}'''_{[n]}\sim_g\mathbf{x}'_{[n]}$ imply $\mathbf{x}''_{[n]}\sim\mathbf{x}_{[n]}$ and $\mathbf{x}'''_{[n]}\sim\mathbf{x}'_{[n]}$ hence in particular $\mathbf{x}'_{[n]}\succ\mathbf{x}_{[n]}$, a contradiction. It follows that $\succcurlyeq\subseteq\succcurlyeq_g$as well. Hence $\succcurlyeq_g=\succcurlyeq$ and the proof is complete. □

Proof of Proposition 3

$\Longleftarrow$ Since we are dealing here exclusively with the additive fragment of elementary arithmetic, and addition (which the 'natural' order $\leq$ over $\mathbb{Z}_+$ relies on for its very definition) is defined by means of the successor function S $:\mathbb{Z}_+\longrightarrow\mathbb{Z}_+$ a few basic points are worth recalling here. Namely, S is a one-to-one function such that every *positive* integer x is the successor of a nonnegative integer, while 0 is not, and of course $\mathrm{S}(0)=1$. Moreover, addition $+$ is defined by the rule: $x+\mathrm{S}(y)=\mathrm{S}(y)+x:=\mathrm{S}(x+y)$, $x+0=0+x:=x$, and the 'natural' order $\leq$ is defined by the rule: for any $x,y\in\mathbb{Z}_+$, $x\leq y$ if there exists $z\in\mathbb{Z}_+$ such that $y=x+z$. Clearly, it follows that addition is *commutative* by definition, and it can be easily checked that it is also *associative*.

From all of the above it follows that $\succcurlyeq^{*av}_{\delta_g}$satisfies AN as a plain consequence of associativity and commutativity of addition, and both S-ANT and RTI as a consequence of the definition of $\leq$ and of associativity and commutativity of addition.

$\Longrightarrow$Now, suppose $\succeq$ is a total preorder over $\mathcal{Z}$ that satisfies AN, S-ANT, RTI. The present proof relies on the definition of a family of auxiliary total preorders over $\mathcal{Z}$ indexed by $[n] \setminus \{1\}$, namely $\{\succeq^m : m = 2, ..., n\}$ defined by the following rule: $\succeq^m$

is such that for any $\boldsymbol{d}, \boldsymbol{d}' \in \mathcal{Z}$ and any subset $M \subseteq [n]$ of cardinality m with $d_h = d'_h$ for every $h \in [n] \setminus M$, $\mathbf{d} \succeq \mathbf{d}'$ if and only if $\sum_{i \in M} d'_i \geq \sum_{i \in M}^n d_i$.

Then, the bulk of the proof amounts to a sort of restricted induction argument on $[n] \setminus \{1\}$ which consists of two steps, namely: (I) $\succeq = \succeq^2$, (II) for any m, $2 \leq m < n$ if $\succeq = \succeq^m$then $\succeq = \succeq^{m+1}$. That is so, because once both (I) and (II) are established one only has to observe that they jointly imply that in particular $\succeq = \succeq^n$. Moreover, as it is easily checked $\succeq^n = \succcurlyeq^{*av}_{\delta_g}$, by definition. Thus, it follows that $\succeq = \succcurlyeq^{*av}_{\delta_g}$as required.

Therefore, we only need to prove that both (I) and (II) hold true.

Step (I). Let us first consider any $i, j \in [n]$, $i \neq j$ and $\boldsymbol{d}, \boldsymbol{d}' \in \mathcal{Z}$ such that $d'_i + d'_j \geq d_i + d_j$, and $d_h = d'_h$ for every $h \in [n], i \neq h \neq j$ (and suppose without loss of generality that $i = 1$ and $j = 2$).

If $\mathbf{d} = \mathbf{d}'$ then of course $\mathbf{d} \sim \mathbf{d}'$ by reflexivity of $\succeq$, so we assume without loss of generality that $\boldsymbol{d} \neq \boldsymbol{d}'$. Five cases are to be considered:

(i) $d'_1 > d_1$ and $d'_2 \geq d_2$ or $d'_1 \geq d_1$ and $d'_2 > d_2$: in this case $\boldsymbol{d} \succ \boldsymbol{d}'$ follows immediately from S-ANT;

(ii) $d'_1 > d_1$, $d_2 > d'_2$ and $d'_1 + d'_2 > d_1 + d_2$: let us first denote by $\mathbf{e}_h := (e_{h1}, ..., e_{hn}) \in \mathbb{Z}^n$, for any $h \in [n]$, the *unit vector* with $e_{hh} = 1$ and $e_{hh'} = 0$ for any $h' \in [n]$, $h' \neq h$. Moreover, let $k_1 := (d'_1 - d_1) > 0$, and $k_2 := (d_2 - d'_2) > 0$, whence $k := (k_1 - k_2) > 0$. Furthermore, AN implies that $\mathbf{e}_h \sim \mathbf{e}_{h'}$ for any $h, h' \in [n]$: hence, in particular, $\mathbf{e}_1 \sim \mathbf{e}_2$. Next, observe that

$\mathbf{d}' = d_1\mathbf{e}_1 + k_1\mathbf{e}_1 + d'_2\mathbf{e}_2$ and $\mathbf{d} = d_1\mathbf{e}_1 + d'_2\mathbf{e}_2 + k_2\mathbf{e}_2$.

Now, RTI implies that $\mathbf{e}_1 + \mathbf{e}_2 \sim \mathbf{e}_2 + \mathbf{e}_2$ and $\mathbf{e}_2 + \mathbf{e}_1 \sim \mathbf{e}_1 + \mathbf{e}_1$ whence, by transitivity of $\succeq$ and commutativity of addition, $\mathbf{e}_1 + \mathbf{e}_2 \sim \mathbf{e}_1 + \mathbf{e}_1 \sim \mathbf{e}_2 + \mathbf{e}_2$. But then, it also follows from a repeated application of RTI and commutativity plus associativity of addition that $\mathbf{d}' \sim d_1\mathbf{e}_1 + d'_2\mathbf{e}_1 + k_2\mathbf{e}_1 + k\mathbf{e}_1$, and $\mathbf{d} \sim d_1\mathbf{e}_1 + d'_2\mathbf{e}_1 + k_2\mathbf{e}_1$. Thus, $\mathbf{d}' \sim \mathbf{d} + k\mathbf{e}_1$, hence $\mathbf{d} \succ \mathbf{d}'$ by transitivity of $\succeq$ since $\mathbf{d} \succ \mathbf{d} + k\mathbf{e}_1$ by S-ANT.

(iii) $d'_i > d_i$, $d_j > d'_j$ and $d'_i + d'_j = d_i + d_j$; by replicating the argument previously used for case (ii) and using the same notation, we obtain the same identities with $k_1 = k_2$ and consequently $k = 0$. It follows that $\mathbf{d} \sim \mathbf{d}'$.

(iv) $d_i > d'_i$, $d'_j > d_j$ and $d'_i + d'_j > d_i + d_j$; by replicating the same argument used for case (ii) and using a similar notation, with $k_1 := (d_1 - d'_1) > 0$, $k_2 := (d'_2 - d_2) > 0$ and $k := (k_2 - k_1) > 0$, we also obtain $\mathbf{d} \succ \mathbf{d}'$.

(v) $d_i > d'_i$, $d'_j > d_j$ and $d'_i + d'_j = d_i + d_j$; by replicating the same argument used for case (iii) and using a similar notation, with $k_1 := (d_1 - d'_1) > 0$, $k_2 := (d'_2 - d_2) > 0$ and $k := (k_2 - k_1) = 0$, we obtain again $\mathbf{d} \sim \mathbf{d}'$.

Thus, we have shown that for any $\boldsymbol{d}, \boldsymbol{d}' \in \mathcal{Z}$ and any $i, j \in [n]$, $i \neq j$ such that $d'_i + d'_j \geq d_i + d_j$, and $d_h = d'_h$ for every $h \in [n], i \neq h \neq j$, it must be the case that $\mathbf{d} \succeq \mathbf{d}'$, and $\mathbf{d} \succ \mathbf{d}'$ if in particular

$d'_i + d'_j > d_i + d_j$.

Conversely, suppose that $\mathbf{d} \succeq \mathbf{d}'$ for some $\boldsymbol{d}, \boldsymbol{d}' \in \mathcal{Z}$ such that for some $i, j \in [n]$, $i \neq j$, both $d_h = d'_h$ for every $h \in [n], i \neq h \neq j$ yet $d_i + d_j > d'_i + d'_j$. Then, it follows from the previous argument that $\mathbf{d}' \succ \mathbf{d}$, a contradiction. Thus, we have in fact shown that for any $\boldsymbol{d}, \boldsymbol{d}' \in \mathcal{Z}$ and any $i, j \in [n]$, $i \neq j$ such that $d_h = d'_h$ for every $h \in [n], i \neq h \neq j$, $\mathbf{d} \succeq \mathbf{d}'$ if and only if $d'_i + d'_j \geq d_i + d_j$ or, equivalently, that $\succeq = \succeq^2$.

Step (II). Suppose that $\succeq = \succeq^m$ for some m, $2 \leq m < n$, and consider any pair of vectors $\boldsymbol{d}, \boldsymbol{d}' \in \mathcal{Z}$ such that $d_h = d'_h$ for every $h \in M' := [n] \setminus \{1, m+1\}$. Then, consider a third vector $\mathbf{d}^* \in \mathcal{Z}$ such that : $d^*_1 + d^*_{m+1} = d_1 + d_{m+1}$, $d^*_{m+1} = d'_{m+1}$, and $d^*_h = d_h$ for all $h \in M'$. Therefore, $\mathbf{d} \sim \mathbf{d}^*$ since $\succeq = \succeq^2$. Moreover, $\succeq = \succeq^m$ implies, by definition, that $\mathbf{d}^* \succeq \mathbf{d}'$ if and only if $\sum_{i=1}^m d'_i \geq \sum_{i=1}^m d^*_i$. It follows that, since by construction $d^*_{m+1} = d'_{m+1}$, $\mathbf{d} \succeq \mathbf{d}'$ if and only if $\sum_{i=1}^{m+1} d'_i \geq \sum_{i=1}^{m+1} d_i$ or, equivalently, $\succeq = \succeq^{m+1}$ as required.

Notice that such a characterization is *tight*, as established by the following three counterexamples:

(i) Let $\succeq^{i^*}$ be a i^*-weakly dictatorial preorder for some $i^* \in [n]$, defined as follows: for any $\mathbf{d} := (d_1 ..., d_n), \mathbf{d}' := (d'_1, ..., d'_n) \in \mathcal{Z}$, $\mathbf{d} \succeq^{\mathbf{i}^*} \mathbf{d}'$ if and only if either $d_{i^*} < d'_{i^*}$ or $[d_{i^*} = d'_{i^*}$ and $\sum_{i=1}^n d_i \leq \sum_{i=1}^n d'_i]$. It can be easily checked that $\succeq^{i^*}$ satisfies both S-ANT and RTI, but violates AN.

(ii) Let $\succeq := \mathcal{Z}^2$ the trivial total preorder over $\mathcal{Z}$ that consists of a unique indifference class. It satisfies AN and RTI, but violates S-ANT.

(iii) Let us now consider a total preorder $\succeq$ over a bounded subset $\mathcal{Z} \subseteq \mathbb{Z}^2_+$ that is defined as follows: for any $\mathbf{d}, \mathbf{d}' \in \mathcal{Z}$, $\mathbf{d} \succeq \mathbf{d}'$ if and only if $f(\mathbf{d}) \leq f(\mathbf{d}')$, where $f : \mathbb{Z}^2_+ \to \mathbb{Z}_+$ is in turn defined by the rule

$$f(\mathbf{d}) = \begin{cases} 2(d_1 + d_2) + 1 & \text{if } d_1 = d_2, \\ 2(d_1 + d_2) & \text{if } d_1 \neq d_2. \end{cases}.$$

Intuitively, $\succeq$ ranks vectors primarily by sum of their components (smaller is better, reflecting a smaller sufficiency-gap), but breaks ties among vectors with equal sums of their components by penalising vectors whose two components coincide. It is immediately checked that $\succeq$ satisfies AN (due to commutativity of addition, and symmetry of equality). Moreover, it also satisfies S-ANT: that is so, because if $\mathbf{d} = \mathbf{d}'$ then $\mathbf{d} \sim \mathbf{d}'$ by definition of $\succcurlyeq$ and there is nothing to prove. If instead $d_i \leq d'_i$ for every $i \in \{1, 2\}$ with $\mathbf{d} \neq \mathbf{d}'$ then $d_1 + d_2 < d'_1 + d'_2$ hence $(d_1 + d_2) + 1 \leq d'_1 + d'_2$. Therefore,

$$f(d) \leq 2(d_1 + d_2) + 1 = 2(d_1 + d_2 + 1) - 1 \leq 2(d'_1 + d'_2) - 1 < 2(d'_1 + d'_2) \leq f(d')$$

which in turn $\mathbf{d} \succ \mathbf{d}'$ as required.

However, $\succeq$ fails to satisfy RTI. To see this, consider $\mathbf{d} = (1, 1)$ and $\mathbf{d}' = (0, 2)$. Then

$$f(1, 1) = 2 \cdot 2 + 1 = 5 \quad \text{and} \quad f(0, 2) = 2 \cdot 2 + 0 = 4,$$

so $\mathbf{d}' \succ \mathbf{d}$ by definition of $\succeq$, whence a fortiori $\mathbf{d}' \succeq \mathbf{d}$. Now add $\mathbf{z} = (2, 0)$ to both profiles:

$\mathbf{d}+\mathbf{z}=(3,1)$ and $\mathbf{d}'+\mathbf{z}=(2,2)$. Then,

$$f(3,1)=2\cdot 4+0=8 \quad \text{and} \quad f(2,2)=2\cdot 4+1=9,$$

so $\mathbf{d}+\mathbf{z}\succ\mathbf{d}'+\mathbf{z}$ hence *not* $\mathbf{d}'+\mathbf{z}\succeq\mathbf{d}+\mathbf{z}$ and RTI is indeed violated. □

Proof of Claim 2.

Let $\succeq$ be a total preorder over $\mathcal{Z}$. Since every permutation $\sigma:[n]\longrightarrow[n]$ decomposes into a finite composition of transpositions $\pi_i:[n]\longrightarrow[n]$, $i=1,...,t$ and $\succcurlyeq$ is transitive, it suffices to show that $\mathbf{d}\sim\mathbf{d}_\pi$ for every $\mathbf{d}\in\mathcal{Z}$ and every transposition π of two indices $h\neq k$. So, fix $\mathbf{d}\in\mathcal{Z}$ and a transposition π of $h\neq k$. If $d_h=d_k$ then $\mathbf{d}_\pi=\mathbf{d}$ and the conclusion is immediate. Assume therefore, without loss of generality, that

$$d_h>d_k. \tag{1}$$

Set $\mathbf{d}':=\mathbf{d}_\pi$, so that $d'_h=d_k$, $d'_k=d_h$, and $d'_i=d_i$ for all $i\notin\{h,k\}$.
Step 1: $d\succeq d'$**.** We verify that the pair $(\mathbf{d},\mathbf{d}')$ satisfies the hypothesis of RHE with h playing the role of the maximiser in $\mathbf{d}$ and k playing the role of the maximiser in $\mathbf{d}'$.

- All components outside $\{h,k\}$ coincide: $d_i=d'_i$ for all $i\notin\{h,k\}$.
- Position h attains the larger value in $\mathbf{d}$: by assumption, $d_h>d_k$.
- Position k attains the larger value in $\mathbf{d}'$: since $d'_k=d_h>d_k=d'_h$.
- The chain of inequalities required by RHE reads $d'_k\geq d_h>d_k\geq d'_h$, i.e. $d_h\geq d_h>d_k\geq d_k$, which holds by construction.

Therefore, RHE yields $\mathbf{d}\succeq\mathbf{d}'$.
Step 2: $d'\succeq d$**.** Apply the same argument to the pair $(\mathbf{d}',\mathbf{d})$, swapping the roles of h and k. Now $d'_k=d_h>d'_h=d_k$, so position k attains the largest value in $\mathbf{d}'$ and position h attains the largest value in $\mathbf{d}$. The hypothesis of RHE is satisfied symmetrically, and RHE yields $\mathbf{d}'\succeq\mathbf{d}$.

Conclusion. From Steps 1 and 2, $\mathbf{d}\succeq\mathbf{d}'$ and $\mathbf{d}'\succeq\mathbf{d}$, hence $\mathbf{d}\sim\mathbf{d}'=\mathbf{d}_\pi$. Since both $\mathbf{d}$ and transposition π of $h\neq k$ were arbitrary and, as previously mentioned, every permutation is a finite composition of transpositions, transitivity of $\succcurlyeq$ extends the result to all permutations. Hence AN holds: notice that, as a result, any total preorder on the capability-type space $\mathbf{X}$ that satisfies ANT and RHE with respect to 'sufficientarian distances' does qualify as a sufficiency-gap preorder as previously defined. □

Proof of Proposition 4

$\Longleftarrow$ It is easily checked that $\succeq^{*l\max}$ satisfies both S-ANT and RHE by definition.

$\Longrightarrow$ We have to prove that if $\widehat{\succeq}$ satisfies S-ANT and RHE then, for any $\mathbf{d},\mathbf{d}'\in\mathcal{Z}$, $\mathbf{d}\widehat{\succeq}\mathbf{d}'$ if and only if $\mathbf{d}\succeq^{*l\max}\mathbf{d}'$. As mentioned above in the text, the proof relies again on a family of auxiliary total preorders $\widehat{\succeq}^m$ over $\mathcal{Z}$ indexed by $m \in [n] \setminus \{1\}$, namely $\left\{\widehat{\succeq}^m : m = 2, ..., n\right\}$ as defined by the rule $\mathbf{d}\widehat{\succeq}^m\mathbf{d}'$ if and only if $\mathbf{d} \succeq^{*l\max} \mathbf{d}'$ for any $M \subseteq [n]$ with $|M| = m$, and any $\boldsymbol{d}, \boldsymbol{d}' \in \mathcal{Z}$ with $d_h = d'_h$ for every $h \in [n] \setminus M$. And, again, the proof itself amounts to a 'restricted' induction argument on $[n] \setminus \{1\}$ which consists of two steps, namely: (I) $\widehat{\succeq} = \widehat{\succeq}^2$, (II) for any m, $2 \leq m < n$ if $\widehat{\succeq} = \widehat{\succeq}^m$then $\succeq = \widehat{\succeq}^{m+1}$.

Step (I). Let us first consider any $i, j \in [n]$, $i \neq j$ and $\mathbf{d}, \mathbf{d}' \in \mathcal{Z}$ such that $d_i \geq d_j$, $d'_i \geq d'_j$ and $d_h = d'_h$ for every $h \in [n], i \neq h \neq j$, and $\mathbf{d}\widehat{\succeq}^2\mathbf{d}'$, i.e. $\mathbf{d} \succeq^{*l\max} \mathbf{d}'$. Suppose also (without any loss of generality since $\widehat{\succeq}$ satisfies AN by Claim 2), that $i = 1$ and $j = 2$. Thus, actually, $d_1 \geq d_2$ and $d'_1 \geq d'_2$. If $\mathbf{d} = \mathbf{d}'$ then of course $\mathbf{d} \sim \mathbf{d}'$ by reflexivity of $\widehat{\succeq}$, so we assume without loss of generality that $\boldsymbol{d} \neq \boldsymbol{d}'$.

If $d_1 = d'_1$, then $d_2 \neq d'_2$ hence $\mathbf{d} \succeq^{*l\max} \mathbf{d}'$ implies $d_2 < d'_2$ which in turn implies $\mathbf{d}\widehat{\succ}\mathbf{d}'$ by S-ANT (hence in particular $\mathbf{d}\widehat{\succeq}\mathbf{d}$) .

If $d_1 < d'_1$ we can have the following three cases:

(1 :) $d'_1 > d_1 \geq d_2 \geq d'_2$ that by RHE entails that $\mathbf{d}\widehat{\succ}\mathbf{d}'$;

(2 :) $d'_1 > d'_2 > d_1 \geq d_2$ that by S-ANT entails again that $\mathbf{d}\widehat{\succ}\mathbf{d}'$;

(3 :) $d'_1 > d_1 \geq d'_2 \geq d_2$, then consider a vector $\mathbf{c} \in \mathcal{Z}$ arranged in nonincreasing order w.l.o.g. by AN (which holds true because it follows from RHE as established by Claim 2 above) differs from d' only in the first two components $\{1, 2\}$, i.e. $\mathbf{c} = (d_1, d'_2, d'_3, ..., d'_n) = (d_1, d'_2, d_3, ..., d_n)$. By the previous argument, since the first components of $\mathbf{d}$ and $\mathbf{c}$ are the same while $d'_2 \geq d_2$, it follows by S-ANT (actually, by ANT) that $\mathbf{d}\widehat{\succeq}\mathbf{c}$. We further observe that $d'_1 > d_1 \geq d'_2 \geq d'_2$, the first inequality by assumption, the second by construction and the third by reflexivity. Hence, by RHE, we have that $\mathbf{c}\widehat{\succeq}\mathbf{d}'$ that together with $\mathbf{d}\widehat{\succeq}\mathbf{c}$ entails by transitivity $\mathbf{d}\widehat{\succeq}\mathbf{d}'$ as required. It follows that $\widehat{\succeq}^2 \subseteq \widehat{\succeq}$.

Conversely, suppose that for some $i, j \in [n]$, $i \neq j$ and $\mathbf{d}, \mathbf{d}' \in \mathcal{Z}$ such that $d_i \geq d_j$, $d'_i \geq d'_j$ and $d_h = d'_h$ for every $h \in [n], i \neq h \neq j$, $\mathbf{d}\widehat{\succeq}\mathbf{d}'$ yet *not* $\mathbf{d}\widehat{\succeq}^2\mathbf{d}'$, namely *not* $\mathbf{d} \succeq^{*l\max} \mathbf{d}'$ which in turn implies by definition that $\mathbf{d}'\succ^{*l\max}\mathbf{d}$. It follows that either $d'_1 < d_1$, or $d'_1 = d_1$ and $d'_2 < d_2$. In any case, it follows by S-ANT that $\mathbf{d}'\widehat{\succ}\mathbf{d}$, a contradiction. As a result, $\widehat{\succeq} \subseteq \widehat{\succeq}^2$ also holds, hence $\widehat{\succeq} = \widehat{\succeq}^2$ as required.

Step (II) Suppose that $\widehat{\succeq} = \widehat{\succeq}^m$ for some m, $2 \leq m < n$. We have to prove that $\widehat{\succeq} = \widehat{\succeq}^{m+1}$.

Suppose that $\widehat{\succeq} = \widehat{\succeq}^m$for some m, $2 \leq m < n$, and consider any pair of vectors $\boldsymbol{d}, \boldsymbol{d}' \in \mathcal{Z}$ such that $d_h = d'_h$ for every $h \in M' := [n] \setminus \{1, m + 1\}$. Then, consider a third vector $\mathbf{d}^* \in \mathcal{Z}$ such that : $\max\left\{d^*_1, d^*_{m+1}\right\} = \max\{d_1, d_{m+1}\}$, and $d^*_h = d_h$ for all $h \in M'$. Therefore, $\mathbf{d}\widehat{\sim}\mathbf{d}^*$ since $\widehat{\succeq} = \widehat{\succeq}^2$. Moreover, $\widehat{\succeq} = \widehat{\succeq}^m$implies, by definition, that $\mathbf{d}^*\ \widehat{\succeq}\mathbf{d}'$ if and only if $\mathbf{d}^* \succeq^{*l\max} \mathbf{d}$. It follows that $\mathbf{d}\ \widehat{\succeq}\mathbf{d}'$ by transitivity or equivalently, since by construction $\mathbf{d}\ \widehat{\succeq}\mathbf{d}'$ if and only if $\mathbf{d} \succeq^{*l\max} \mathbf{d}'$ that $\succeq = \succeq^{m+1}$as required.

The characterization is also tight. Indeed, it is easily checked that the trivial preorder $\succeq := \mathcal{Z}^2$ satisfies RHE but violates S-ANT. Conversely the min-average preorder $\succeq^{*av}$does satisfy S-ANT but violates RHE (to see this, consider $\mathbf{d} = (10, 9), \mathbf{d}' = (2, 11)$: clearly $d'_2 > d_1 > d_2 > d'_1$, hence

RHE would require $\mathbf{d} \succ \mathbf{d}'$. Yet $\mathbf{d}' \succ^{*av} \mathbf{d}$). $\square$

Proof of Lemma 1

The proof of Lemma 1 consists of two parts.

(a) The first and most important one is the proof that $(\mathcal{A}_{\mathbf{X}}, \gtrapprox)$ is a distributive lattice, which is indeed a classic result due to Dilworth (1960), namely

- Let $\mathbf{Y} = (Y, \leqslant)$ be a finite partially ordered set, $\mathcal{A}_{\mathbf{Y}}$ the set of all *antichains* of $\mathbf{Y}$ (i.e., sets of mutually $\leqslant$-incomparable elements of Y), and $\gtrapprox$ the binary relation on $\mathcal{A}_{\mathbf{Y}}$ defined by the following rule: for any $\mathcal{X} = \{x_1, ..., x_k\}$, $\mathcal{Y} = \{y_1, ..., y_h\} \in \mathcal{A}_{\mathbf{Y}}$, $\mathcal{X} \gtrapprox \mathcal{Y}$ if and only if for every $x_i \in \mathcal{X}$ there exists $y_j \in \mathcal{Y}$ such that $y_j \leqslant x_i$. Then, $(\mathcal{A}_{\mathbf{Y}}, \gtrapprox)$ is a distributive lattice.

(b) $(\mathcal{A}_{\mathbf{X}}, \gtrapprox)$ is isomorphic to the lattice $(\mathcal{S}(\mathbf{X}^n), \leqslant)$ of sufficientarian BGFs on $\mathbf{X}$.

The proof of part (a) to be presented below is *not* the original one due to Dilworth (1960), but rather a *dualized* version of the proof proposed by Anderson (1987) focussing on the correspondence between antichains and *order ideals* (or downward closed sets) as opposed to the correspondence between antichains and *order filters* (or upward closed sets) of $\mathbf{Y}$, which is the relevant one in our case. Thus, for the sake of both convenience and completeness, we report our 'dualized' version of Anderson's proof in some detail below. To begin with, recall that a set $F \subseteq Y$ is an *order filter* of $\mathbf{Y} = (Y, \leqslant)$ -also written $F \in \mathcal{F}_{\mathbf{Y}}$- if and only if [for every $x, y \in Y$, if $x \in F$ and $x \leqslant y$ then $y \in F$].

Part (a) Then, the proof consists in establishing the validity of the claims attached to the following five steps.

Step (i) $(\mathcal{A}_{\mathbf{Y}}, \gtrapprox)$ is a partially ordered set (or poset). It must be checked that $\gtrapprox$ is reflexive, antisymmetric and transitive. Clearly, for any $\mathcal{X} \in \mathcal{A}_{\mathbf{Y}}$, $\mathcal{X} \gtrapprox \mathcal{X}$ follows immediately, by definition, from reflexivity of $\leqslant$: thus, *reflexivity* of $\gtrapprox$ holds. Moreover, consider any pair of antichains $\mathcal{X}, \mathcal{Y} \in \mathcal{A}_{\mathbf{Y}}$ such that $\mathcal{X} \gtrapprox \mathcal{Y}$ and $\mathcal{Y} \gtrapprox \mathcal{X}$: by definition, for every $x \in \mathcal{X}$ there exists $y \in \mathcal{Y}$ such that $y \leqslant x$, and for every $y' \in \mathcal{Y}$ there exists $x' \in \mathcal{X}$ such that $x' \leqslant y'$. Now, consider the first pair with $x \in \mathcal{X}$ arbitrarily chosen and $y \in \mathcal{Y}$ with $y \leqslant x$. Since $\mathcal{Y} \gtrapprox \mathcal{X}$, there exists $x' \in \mathcal{X}$ such that $x' \leqslant y$. But then, both $y \leqslant x$ and $x' \leqslant y$. Thus, by transitivity of $\leqslant$, $x' \leqslant x$ while both of them belong to antichain $\mathcal{X}$. It follows that $x = x'$, that in turn implies that $y = x$. Therefore, $\mathcal{X} \subseteq \mathcal{Y}$. But then, a similar argument applied to the second pair x', y' mentioned above, with $y' \in \mathcal{Y}$ arbitrarily chosen, and $x' \in \mathcal{X}$ such that $x' \leqslant y'$, establishes that $\mathcal{Y} \subseteq \mathcal{X}$. Thus, $\mathcal{X} = \mathcal{Y}$ and *antisymmetry* of $\gtrapprox$ holds. Finally, consider any $\mathcal{X}$,$\mathcal{Y}$,$\mathcal{Z} \in \mathcal{A}_{\mathbf{Y}}$ such that $\mathcal{X} \gtrapprox \mathcal{Y}$ and $\mathcal{Y} \gtrapprox \mathcal{Z}$. Next, consider an arbitrary $x \in \mathcal{X}$. Since $\mathcal{X} \gtrapprox \mathcal{Y}$, there exists $y \in \mathcal{Y}$ such that $y \leqslant x$: Also, since $\mathcal{Y} \gtrapprox \mathcal{Z}$, there exists $z \in \mathcal{Z}$ such that $z \leqslant y$. But then, $z \leqslant x$ holds by transitivity of $\leqslant$: it follows that *transitivity* of $\gtrapprox$ also holds. Hence $(\mathcal{A}_{\mathbf{Y}}, \gtrapprox)$ is indeed a poset, as required.

Step (ii) For any antichain $\mathcal{Y} \in (\mathcal{A}_{\mathbf{Y}}, \gtrapprox)$ the set

$\widehat{\mathcal{Y}} := \{y \in Y :$ there exists an $x \in \mathcal{Y}$ such that $x \leqslant y\} \in \mathcal{F}_{\mathbf{Y}}$, i.e., is an order filter of $\mathbf{Y}$. That is immediate, by definition. Indeed, let $\mathcal{Y} \in \mathcal{A}_{\mathbf{Y}}$ and $x, y \in Y$ such that $x \in \widehat{\mathcal{Y}}$ and $x \leqslant y$. Then, by definition, there exists $x' \in \mathcal{Y}$ such that $x' \leqslant x$. Therefore by transitivity $x' \leqslant y$ as well, and by definition $y \in \widehat{\mathcal{Y}}$. Hence $\widehat{\mathcal{Y}}$ is indeed an order filter of $\mathbf{Y}$.

Step (iii) For any order filter $F \in \mathcal{F}_{\mathbf{Y}}$ the set

$F_{\min} := \{y \in F : \text{for any } x \in Y,\ x \leqslant y \text{ only if } x = y\} \in \mathcal{A}_{\mathbf{Y}}$, i.e., is an antichain of $\mathbf{Y}$. Hence, in particular, for any order filter $F \in \mathcal{F}_{\mathbf{Y}}$ there exists an antichain $\mathcal{Y} :=F_{\min} \in \mathcal{A}_{\mathbf{Y}}$ such that $F = \widehat{\mathcal{Y}}$, and conversely if $F = \widehat{\mathcal{Y}}$ for some $\mathcal{Y} \in \mathcal{A}_{\mathbf{Y}}$ then $\mathcal{Y} = F_{\min}$. That is also immediate, by definition. To see this, consider any two *distinct* $x, y \in F_{\min}$.Clearly, neither $x \leqslant y$ nor $y \leqslant x$ is the case, because each one of those inequalities implies $x = y$, a contradiction. Hence, $F_{\min}$ is in fact an antichain of $\mathbf{Y}$. Moreover, for any $F \in \mathcal{F}_{\mathbf{Y}}$, $F = \widehat{F}_{\min}$ and for any $\mathcal{Y} \in \mathcal{A}_{\mathbf{Y}}$, $F = \widehat{\mathcal{Y}}$ only if $\mathcal{Y} = F_{\min}$ by construction.

Step (iv) For any pair of antichains $\mathcal{X}, \mathcal{Y} \in \mathcal{A}_{\mathbf{Y}}$, [$\mathcal{X} \gtrapprox \mathcal{Y}$ if and only if $\widehat{\mathcal{X}} \subseteq \widehat{\mathcal{Y}}$] and [$\mathcal{X} \approx \mathcal{Y}$ if and only if $\widehat{\mathcal{X}} = \widehat{\mathcal{Y}}$]. Suppose that $\mathcal{X} \gtrapprox \mathcal{Y}$ and $z \in \widehat{\mathcal{X}}$ (with $\mathcal{Y} = \widehat{\mathcal{Y}}_{\min}$, and $\mathcal{X} = \widehat{\mathcal{X}}_{\min}$, by construction, as shown under step (iii)). Then, by definition of $\widehat{\mathcal{X}}$, there exists $x \in \mathcal{X}$ such that $x \leqslant z$. Now, $\mathcal{X} \gtrapprox \mathcal{Y}$ implies that for any $x \in \mathcal{X}$ there exists $y' \in \mathcal{Y}$ such that $y' \leqslant x$. Hence $y' \leqslant z$ by transitivity of $\leqslant$, and $z \in \widehat{\mathcal{Y}}$ which in turn implies that $\widehat{\mathcal{X}} \subseteq \widehat{\mathcal{Y}}$. Conversely, suppose that $\widehat{\mathcal{X}} \subseteq \widehat{\mathcal{Y}}$. Then, for every $x \in \widehat{\mathcal{X}}$ there exists $y \in \mathcal{Y}$ such that $x \leqslant y$ (since $x \in \widehat{\mathcal{Y}}$ as well). Thus, in particular, for every $x \in \widehat{\mathcal{X}}_{\min} = \mathcal{X}$ there exists $y \in \mathcal{Y}$ such that $x \leqslant y$. It follows that, by definition, $\mathcal{X} \gtrapprox \mathcal{Y}$. Finally, suppose that $\mathcal{X} \approx \mathcal{Y}$, i.e., both $\mathcal{X} \gtrapprox \mathcal{Y}$ and $\mathcal{Y} \gtrapprox \mathcal{X}$. Then, it follows from the previous part of the present step that both $\widehat{\mathcal{X}} \subseteq \widehat{\mathcal{Y}}$ and $\widehat{\mathcal{Y}} \subseteq \widehat{\mathcal{X}}$ hold. Therefore, $\widehat{\mathcal{X}} = \widehat{\mathcal{Y}}$. Conversely, by the very same argument, if $\widehat{\mathcal{X}} = \widehat{\mathcal{Y}}$ then $\mathcal{X} \approx \mathcal{Y}$ as required.

Step (v): $(\mathcal{F}_{\mathbf{Y}}, \subseteq)$ is a distributive lattice and is isomorphic to $(\mathcal{A}_{\mathbf{Y}}, \gtrapprox)$. Hence $(\mathcal{A}_{\mathbf{Y}}, \gtrapprox)$ is also a distributive lattice. The previous steps (ii),(iii),(iv) jointly imply that $(\mathcal{F}_{\mathbf{Y}}, \subseteq)$ is isomorphic to $(\mathcal{A}_{\mathbf{Y}}, \gtrapprox)$. Now, consider an arbitrary pair of order filters $F, F' \in \mathcal{F}_{\mathbf{Y}}$ and their set-theoretic intersection $F \cap F'$, and union $F \cup F'$. Let $x, y \in Y$ be such that $x \leqslant y$ and $x \in F \cap F'$. Then, by definition, there exist $z \in F$ and $z' \in F'$ such that $z \leqslant x$ and $z' \leqslant x$. It follows that both $z \leqslant y$ and $z' \leqslant y$, by transitivity of $\leqslant$. Thus, $y \in F \cap F'$. It follows that $F \cap F' \in \mathcal{F}_{\mathbf{Y}}$, i.e., is an order filter of $\mathbf{Y}$.

Moreover, let $x, y \in Y$ be such that $x \leqslant y$ and $x \in F \cup F'$. Then, by definition, there exists either a $z \in F$ such that $z \leqslant x$ or a $z' \in F'$ such that $z' \leqslant x$ (or both of them). Suppose then without any loss of generality that there exists a $z \in F$ such that $z \leqslant x$. It follows again that $z \leqslant y$, by transitivity of $\leqslant$. Thus, $y \in F \cup F'$. It follows that $F \cup F' \in \mathcal{F}_{\mathbf{Y}}$, i.e., it is also an order filter of $\mathbf{Y}$. But then, $(\mathcal{F}_{\mathbf{Y}}, \subseteq)$ is a *lattice* with a well-defined g.l.b or *meet* operation $\wedge := \cap$, and a well-defined l.u.b. or *join* operation $\vee := \cup$, and the (mutual) distributivity laws satisfied by set-theoretic intersection $\cap$ and union $\cup$ imply that $(\mathcal{F}_{\mathbf{Y}}, \subseteq)$ is a *distributive* lattice (and the same holds for $(\mathcal{A}_{\mathbf{Y}}, \gtrapprox)$).

Part (b): Let us now consider our case where $\mathbf{Y} = \mathbf{X}$. To prove that $(\mathcal{A}_{\mathbf{X}}, \gtrapprox)$ is isomorphic to $\mathcal{S}(\mathbf{X}^n)$, just consider the function $\varphi : \mathcal{S}(\mathbf{X}^n) \longrightarrow \mathcal{A}_{\mathbf{X}}$ defined as follows: for any $g \in \mathcal{S}(\mathbf{X}^n)$, $\varphi(g) := \mathcal{X}(g)$. Then, it is easily checked that φ is a latticial isomorphism. □

Proof of Proposition 5

By Lemma 1 $(\mathcal{A}_{\mathbf{X}}, \precapprox)$ is a distributive lattice that is isomorphic to $(\mathcal{S}(\mathbf{X}^n), \leqslant)$. Since $(\mathcal{A}_{\mathbf{X}}, \precapprox)$ is a distributive lattice, two important corollaries follow, namely

(I) A *metric* betweenness B_d (which as observed previously in the text is also a median betweenness) can be defined on $(\mathcal{A}_{\mathbf{X}}, \precapprox)$, and a natural domain of preference preorders $\succeq$ on $\mathcal{A}_{\mathbf{X}}$ with a unique top antichain that are single-peaked with respect to B_d (namely, for any $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \in B$ if $\mathcal{X} = top(\succeq) \neq \mathcal{Z}$ then $\mathcal{Y} \succeq \mathcal{Z}$) (see Savaglio and Vannucci (2019)).

(II) By Lemma 2, there exists a class of (lattice-polynomial) anonymous and idempotent aggregation rules $F : (\mathcal{A}_{\mathbf{X}})^n \longrightarrow \mathcal{A}_{\mathbf{X}}$ on the preference domain $\mathcal{D}_{B_d}$ as defined above (see Savaglio and Vannucci (2019) and Vannucci (2019)). If n is odd, that class includes the simple majority aggregation rule F^{maj} that is defined as follows: for any profile of antichains $\mathcal{X}_{[n]} = (\mathcal{X}_1, ..., \mathcal{X}_n) \in (\mathcal{A}_{\mathbf{X}})^n$, $F^{maj}(\mathcal{X}_{[n]}) =_{S \in \mathcal{W}^{maj}} {}_{i \in S} \mathcal{X}_i$, where $\mathcal{W}^{maj} := \left\{ S \subseteq N : |S| \geq \left\lfloor \frac{|N|+2}{2} \right\rfloor \right\}$. Thus, the protocol $\varphi_{[n]}^{-1} \circ F^{maj} \circ \varphi^{-1}$ is a strategy-proof implementation of the aggregation rule $G : (\mathcal{S}(\mathbf{X}^n)\ )^n \longrightarrow \mathcal{S}(\mathbf{X}^n)$ where $\varphi_{[n]} := (\varphi, ..., \varphi)$. $\square$