Introduction

The equilibrium concept developed by Cournot (1838) is commonly employed in the analysis of oligopolistic markets. In its most traditional and basic formulation this concept is aimed at identifying the output price and the output quantities resulting from the decentralized choices of a finite number of firms producing the same commodity. When regarded as a prediction of a complete theory on firms' behavior, a Cournot equilibrium should be unique: failing uniqueness, the realization of a Cournot equilibrium is determined by something external to the theory, which, therefore, turns out to be incomplete. The purpose of the present work is to establish necessary and sufficient conditions for the existence of a unique Cournot equilibrium within classes of oligopolies.

The first difficulty one encounters when examining a problem of necessary and sufficient conditions is the formulation of the problem. The notions of an inverse demand function and of a cost function, constitutive elements of the concept of a Cournot equilibrium, are here reviewed so as to obviate some questionable aspects of the original Cournotian construction. By relying upon the separation between the inverse demand function and the cost function an essential class of oligopolies is selected. Four necessary conditions for the existence of a unique Cournot equilibrium are established on this class. It is then shown that the same conditions suffice for the existence of a unique Cournot equilibrium in a much wider class of oligopolies which is very significant from an economic standpoint. Thus are obtained two propositions on necessary and sufficient conditions for the existence of a unique Cournot equilibrium.

The sequel is organized as follows. Section 1 describes the economic context underlying the analysis elaborated in the succeeding sections and provides the formal definition of an oligopoly. Section 2 discusses the Cournot equilibrium concept dwelling on the importance of uniqueness. Section 3, in light of the formulation advanced in Section 1, sets the terms of the problem of necessary conditions for the existence of a unique Cournot equilibrium and establishes a first proposition on necessary and sufficient conditions for the existence of a unique Cournot equilibrium. Section 4 identifies a class of oligopolies satisfying a certain closure property, explains why this property is desirable from an economic standpoint and establishes, within this latter class, a second proposition on necessary and sufficient conditions for the existence of a unique Cournot equilibrium. Section 5, by drawing a comparison with the scanty literature on this subject, summarizes the results achieved and provides a justification for the structure of the problem of necessary and sufficient conditions as here set forth. Almost all proofs are contained in the appendices that follow Section 3 and Section 4.

This work should be read in full since its various parts cannot be disjoined, yet by reading Section 5 and skimming the rest one may gain an insight into the results obtained.

Notational and Terminological Conventions

This work employs only elementary tools of mathematics. Our notation and terminology are standard. Here we disambiguate only those conventions which may be misinterpreted.

$A \subseteq B$ entails that if $x \in A$ then $x \in B$.

$A \subset B$ entails that $A \subseteq B$ and $A \neq B$.

$\mathbb{R}$ denotes the set of all real numbers.

$\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$.

$\mathbb{R}_- := \{x \in \mathbb{R} : x \leq 0\}$.

$\mathbb{R}_{++} := \{x \in \mathbb{R} : x > 0\}$.

$\mathbb{R}_{--} := \{x \in \mathbb{R} : x \leq 0\}$.

$x \in \mathbb{R}$ is positive if $x > 0$.

$x \in \mathbb{R}$ is negative if $x < 0$.

$x \in \mathbb{R}$ is nonnegative if $x \geq 0$.

$x \in \mathbb{R}$ is nonpositive if $x \leq 0$.

Let A and B be two nonempty sets such that $\emptyset \notin B$:

-a single-valued function $f : A \to B$ associates with each $x \in A$ one and only one element of B,

-a multi-valued function $f : A \to B$ associates with each $x \in A$ at least one element of B.

When unspecified it is understood that $f : A \to B$ is a single-valued function and that "function" stands for single-valued function.

Let $\emptyset \neq A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. f is:

-increasing if $\forall x, y \in A$, $x > y$ entails that $f(x) > f(y)$,

-decreasing if $\forall x, y \in A$, $x > y$ entails that $f(x) < f(y)$,

-nondecreasing if $\forall x, y \in A$, $x > y$ entails that $f(x) \geq f(y)$,

-nonincreasing if $\forall x, y \in A$, $x > y$ entails that $f(x) \leq f(y)$.

Let $f : \mathbb{R} \to \mathbb{R}$, $\emptyset \neq A \subset \mathbb{R}$ and $g : A \to \mathbb{R}$:

-f is linear if, $\forall x \in \mathbb{R}$, $f(x) = x(f(1) - f(0))$,

-g is linear if there exists an extension of g from A to $\mathbb{R}$ which is linear.

Let $f : A \to B$ and $C \subseteq A$. C is closed under f if $\forall x \in C$ $f(x) \in C$.

Let $f : A \to \mathbb{R}$ and let f possess a maximum over $B \subseteq A$:

-$\max\limits_{z \in B} f(z)$ denotes the maximum of f over B,

-$\min\limits_{z \in B} -f(z)$ denotes the minimum of $-f$ over B,

-$\arg\max\limits_{x \in B} f(x) := \{\, x \in B : f(x) = \max\limits_{z \in B} f(z) \,\}$,

-$\arg\min\limits_{x \in B} -f(x) := \arg\max\limits_{x \in B} f(x)$.

Let $A \neq \emptyset$ be a set of reals:

-$f'(x)$ denotes the derivative of $f : A \to \mathbb{R}$ at x,

-$g'_+(x)$ denotes the right-hand derivative of $g : A \to \mathbb{R}$ at x,

-$h'_-(x)$ denotes the left-hand derivative of $h : A \to \mathbb{R}$ at x.

1 - Context and Definitions

While turning over the leaves of an economics textbook we may come across a definition of oligopoly that goes something like this: An oligopoly is a market form in which a small number of firms compete in the production of commodities. *Prima facie* we may be satisfied with this definition, but a closer examination would lead us to wonder what it really means and specifies[1]. Phrases like "small number" or "market form" seem quite vague and the observation of the fact that the competition of firms has to do with production is not so illuminating.

This writing focuses on a particular model of competition among producers devised by Cournot in 1838[2]. This section circumscribes the context of the problems to be addressed in the present work, and the definitions, as here formulated, pertain only to our specific subject-matter. Ineffectual generalizations will be avoided.

We are concerned with a finite set of single-output firms independently producing the same good. The market for this good opens periodically and closes before the end of each period. Production takes place during the market closure, and, at each opening, the overall quantity produced in the preceding period is supplied to the market. Henceforth, unless otherwise specified, we shall focus on the lapse of time occurring between a closing and the succeeding opening.

The firms in question have already incurred fixed costs. These costs are required to set up production and they cannot be recovered over the time period under consideration here, hence they are to be considered sunk. This also entails that, over the same time period, the firms' technological characteristics cannot be modified. The productive capacity of a firm is here defined as the set of its possible production levels and is therefore represented as a subset of the real line. We denote by K_i the productive capacity of a generic firm i. In our context the presence of sunk costs explains the fixity in the number of firms: were sunk costs absent the number of firms need not be fixed.

These sunk costs, mirroring a past choice of adoption of a specific technology, determine the productive capacity and the cost function of each firm. The cost function of a firm assigns to each feasible output level its least production cost. The productive capacity of a firm is the domain of the cost function of that firm. We denote by c_i the cost function of a generic firm i. It should be clear that c_i is a function from K_i to $\mathbb{R}$.

We assume that each productive capacity contains the null output level and at least one positive output level. The former assumption follows from the idea that inaction is possible, while the latter makes the choice problem nontrivial. We also assume that each productive capacity is a closed convex subset of the

[1] The reader may enjoy Edward H. Chamberlin's "On the Origin of Oligopoly", *The Economic Journal* (1957), in which the writer claims the authorship of the term "oligopoly".

[2] Antoine Augustine Cournot, *Recherches sur les principes mathématiques de la théorie des richesses*, 1838.

nonnegative real half-line. These assumptions entail that if two output levels are feasible then every intermediate level is feasible and that the productive capacity contains its boundary.

We assume that, at each opening, the aggregate production is supplied to the market[3], where it is sold at the greatest monetary value at which it is entirely demanded. When the aggregate production is positive the monetary value of the sale is apportioned to the firms proportionally to the quantities produced, and when the aggregate production is null each firm receives nothing from the sale.

The reader acquainted with the literature on Cournot equilibrium might be surprised not to find the usual construction based on the inverse of the Walrasian demand multi-valued function. That construction resorts to the idea that the firms announce the same market-clearing price. Now, if the sale mechanism were centralized our firms would not set any price, and if the sale mechanism were decentralized our firms could freely set the prices, which, in principle, may also differ from one another. In Cournot's model this last possibility is not even taken into account since there exists a single price for the output, whatever the seller. For this reason, and unlike Cournot[4], we shall not assume the firms to set prices; yet we shall not assume price-taking behavior either. We shall simply assume the existence of a centralized sale mechanism according to which the revenue coming from the sale of the entire aggregate production is shared by the firms. At the end of this section we shall introduce a function, named price function, which can be mathematically treated as an inverse demand function. The reader may be surprised to see coming back trough the window what has been thrown out through the door; actually what will be coming back is something different but allows us to preserve the usual form of the model, so that replacing the phrase "price function" with "inverse demand function" all the results of this work continue to hold even when the concept of an inverse demand function is maintained.

The multi-valued aggregate revenue function is a multi-valued function $r : \mathbb{R}_+ \to \mathbb{R}_+$ associating with each nonnegative level of aggregate supply all the possible monetary values at which that level of aggregate supply is entirely demanded. Thus the multi-valued aggregate revenue function is a primitive concept, and if it is true that it is not based on the maximizing behavior of the agents of the economy, of which our firms are integral part, it is also true that this lack is not felt, since we are not considering the economy as a whole. In particular, the introduction of extrinsic concepts, such as that of a Wal-

[3] We are implicitly assuming that the good is supplied only by the producers.

[4] Taking into consideration a couple of firms managed by two different proprietors Cournot writes: "Proprietor 1 can have no direct influence on the determination of D_2 [the production level of the second firm]: all that he can do, when D_2 has been determined by proprietor 2, is to choose for D_1 [the production level of the first firm] the value which is best for him. This he will be able to accomplish by properly adjusting his price". The original Cournotian price formation mechanism is in a sense inconsistent with the rest of the model in which only the levels of production are treated as independent variables. The murkiness in the deserved distinction between independent and dependent variables has given rise to various criticisms, among the most notable ones that of Bertand, that of Edgeworth and that of Pareto.

rasian demand function, seems to the writer just a straining which explains little but burdens our model with unnecessary assumptions. The definitions provided so far suffice for the definition of an oligopoly. An oligopoly is a 4-tuple $(r, n, (K_1, .., K_n), (c_1, .., c_n))$, where r is a multi-valued aggregate revenue function for a good γ, n is a positive integer number representing the number of firms producing γ, K_i is firm i's productive capacity, $c_i : K_i \to \mathbb{R}$ is firm i's cost function, $(K_1, .., K_n)$ is an n-tuple of productive capacities and $(c_1, .., c_n)$ is a n-tuple of cost functions.

In the following we shall assume that $r(0) := \{0\}$, that r is compact-valued, that each level of aggregate supply x is sold at $\max r(x)$ and that the function $\bar{r} : \mathbb{R}_+ \to \mathbb{R}_+$, $\bar{r} : x \mapsto \max r(x)$ is continuous over $\mathbb{R}_+$. Let us examine these assumptions one by one. The fact that a null level of supply is associated only with a null revenue seems hardly questionable. The idea that r is compact-valued entails that the image of r, at every point of its domain, is a closed and bounded set, in particular is bounded since we imagine that for every level of supply there exists a positive level of possible revenues which cannot be overcome. The previous assumption allows $\bar{r}$ to be defined over $\mathbb{R}_+$. Also in this case it seems quite natural to assume that each quantity is sold at the greatest monetary value among the possible ones. Continuity of $\bar{r}$ derives from the idea that there are no jumps in the values attained by this function.

Before providing the definition of a price function we can introduce three more assumptions ruling out a few trifling cases. We assume that $\bar{r}$ is null for at least one value of $\mathbb{R}_{++}$, that $\bar{r}$ is locally Lipschitz continuous at 0 [5] and that $\liminf_{x \to 0^+} \frac{\bar{r}(x)}{x} > 0$. The first assumption derives from the idea that there exists at least one level of supply, exceeding physically reasonable sizes, at which the commodity ceases to be an economic good. Local Lipschitz continuity of $\bar{r}$ at 0 entails that $\limsup_{x \to 0^+} \frac{\bar{r}(x)}{x}$ is a real number[6]. There is no particular economic reason to discard the cases $\limsup_{x \to 0^+} \frac{\bar{r}(x)}{x} = +\infty$ and $\liminf_{x \to 0^+} \frac{\bar{r}(x)}{x} = 0$, yet, to the writer, there is no particular reason to think that they are economically significant: they are discarded for analytical convenience.

Given the previous definitions and assumptions we can now derive from the multi-valued aggregate revenue function r a function p, named price function, which will be extremely convenient in the next sections. The price function is a function $p : \mathbb{R}_+ \to \mathbb{R}_+$ such that $p(x) := \frac{\max r(x)}{x}$ for every $x > 0$ and $p(0) := \limsup_{x \to 0^+} p(x)$. Hence, for every positive x, $p(x)$ is, very simply, the greatest unitary value of the aggregate revenue at x. Because of our assumption on r, p is continuous over $\mathbb{R}_{++}$ and bounded over bounded sets, and the set $\{x \in \mathbb{R}_{++} : p(x) = 0\}$ has a minimum. We have explained how, under the conditions mentioned above, to derive the price function from an aggregate revenue

[5] Let $D \subseteq \mathbb{R}$ and $x^\circ \in D$. $f : D \to \mathbb{R}$ is Lipschitz continuous at x° if there exist a number $k \in \mathbb{R}_+$ such that, $\forall x \in D$, $|f(x) - f(x^\circ)| \leq k|x - x^\circ|$.

$f : D \to \mathbb{R}$ is locally Lipschitz continuous at x° if there exist a number $k \in \mathbb{R}_+$ and an open bounded interval I containing x° such that, $\forall x \in D \cap I$, $|f(x) - f(x^\circ)| \leq k|x - x^\circ|$.

[6] For instance, the price function $\bar{r} : x \mapsto \sqrt{x}$ is not locally Lipschitz continuous at 0.

function; it is straightforward to verify that $p(x)x = \bar{r}(x) = \max r(x)$ for every nonnegative aggregate supply x.

We can now give the definition of an oligopoly which, unless otherwise specified, will be valid throughout this work. On this definition we shall construct the classes of oligopolies which will be the basis of our study in the sequel.

Definition 1 *Let $p : \mathbb{R}_+ \to \mathbb{R}_+$ be a price function. We define an oligopoly $(p, n, (K_1, .., K_n), (c_1, .., c_n))$ as a 4-tuple of the following components: a price function p; a positive integer number n of firms; an n-tuple of productive capacities $(K_1, ..., K_n)$ such that K_i, for $i = 1, .., n$, is a closed real interval with minimum 0 and nonempty interior; an n-tuple of cost functions $(c_1, ..., c_n)$ such that $c_i : K_i \to \mathbb{R}_+$, for $i = 1, .., n$.*[7]

Henceforth we shall assume that all of the firms in an oligopoly mutually know the structure of the oligopoly.

Definition 2 *Given a price function p, we define O^p the set of all oligopolies $(p, n, (K_1, .., K_n), (c_1, .., c_n))$ with price function p. We define O^p_{lin} the set of all oligopolies in O^p with increasing and linear cost functions. We define $O^p_{lin(2)}$ the set of all oligopolies in O^p_{lin} with at least 2 firms. We define O^p_{cvx} the set of all oligopolies in O^p with increasing, continuous and convex cost functions null at 0. We define O^p_{inc} the set of all oligopolies in O^p with increasing cost functions.*

The reason why the structure of these classes of oligopolies is crucial for our analysis will be clear in the next sections, now a justification is given for regarding convex cost functions and linear cost functions as the most significant classes of cost functions within the context so far set forth.

Let us consider an oligopoly with n firms, and let us assume that the technology of each firm $i = 1, .., n$ is described by the usual production function f_i[8], from a convex closed set B_i, such that $\mathbf{0} \in B_i \subseteq \mathbb{R}^{m_i}_+$ and $B_i \cap \mathbb{R}^{m_i}_{++} \neq \emptyset$, onto the productive capacity K_i. Let us now assume that f_i is continuous, concave and null at the origin. These latter three assumptions embody the idea of nonincreasing returns to scale, which seems a natural assumption when, as in our case, fixed sunk costs have already been incurred and fixed nonsunk costs are null. If input prices are positive and fixed then we have that c_i is a convex, increasing and continuous cost functions null at 0; obviously the domain of c_i is K_i. A particular but significant case is when, in addition to the previous conditions, B_i coincides with $\left\{B_i + \mathbb{R}^{m_i}_-\right\} \cap \mathbb{R}^{m_i}_+$ and f_i can be extended to $\mathbb{R}^{m_i}_+$ as a production function of the Leontief type $f^{ext}_i : \mathbb{R}^{m_i}_+ \to \mathbb{R}_+$, $f^{ext}_i : (x_1, .., x_{m_i}) \mapsto \min\{a_1 x_1, ..., a_{m_i} x_{m_i}\}$. f^{ext}_i captures the idea of constant returns to scale and entails the existence of a single optimal ratio of inputs, and it is perhaps

[7] In this definition it is understood that in each oligopoly the domains of the cost functions and of the price function are not sets of pure numbers but sets of quantities with the same physical unit of measurement.

[8] In our case f_i associates with each vector of variable input quantities $\mathbf{x}$ the maximum amount of output that can be produced with $\mathbf{x}$.

the simplest kind of technology one can imagine. When such an extension is possible the cost function associated with f_i is also linear.

In the previous reasoning we have assumed that for each input there exists a unique positive price and that input prices are fixed. The existence of a unique price, sometimes justified by the so-called "law of one price", is here only a convenient and arbitrary assumption. The fixity of input prices embodies the idea that the formation of input prices is not affected by the aggregate demand of the firms of the oligopoly, that is, the influence exerted on input prices by the firms of our oligopoly, whatever their aggregate demand for inputs, is negligible with respect to the price levels that would occur if that oligopoly were not present. This assumption is particularly strong when B_i is not bounded: in this case it is difficult to admit that, whatever the aggregate demand, the effect on input prices is negligible. For this reason the case of bounded productive capacities[9] seems more acceptable than that of unbounded productive capacities.

We can now provide the definition of a profit function. We shall denote by π_i the profit function of firm i. Given an oligopoly $(p, n, (K_1, .., K_n), (c_1, .., c_n))$, π_i associates with every vector $(q_1, .., q_n) \in \prod_{i=1}^n K_i$ the monetary value of firm i's revenue at $(q_1, .., q_n)$, which is a share $p(q_1 + .. + q_n) q_i$ of the aggregate revenue $p(q_1 + .. + q_n)(q_1 + .. + q_n)$, minus the production cost of q_i for firm i, that is $c_i(q_i)$. More formally:

Definition 3 *Given an oligopoly $(p, n, (K_1, .., K_n), (c_1, .., c_n))$, the profit function of firm i $(i = 1, .., n)$ is:*

$$\pi_i : \prod_{i=1}^n K_i \to \mathbb{R}, \quad \pi_i : (q_1, .., q_n) \mapsto p(q_1 + .. + q_n) q_i - c_i(q_i).$$

2 - Cournot "Equilibrium"

Thus far we have defined the concept of an oligopoly but nothing has been said about firms' behavior. This we shall do here by providing a definition of a Cournot equilibrium and an interpretation thereof.

Given an oligopoly $o = (p, n, (K_1, .., K_n), (c_1, .., c_n))$ a Cournot equilibrium is a vector $\mathbf{e} \in \prod_{i=1}^n K_i$ such that e_i maximizes each firm i's profit function when the overall output of the other firms is $\sum_{j \neq i} e_j$. More formally:

Definition 4 *Let $o := (p, n, (K_1, .., K_n), (c_1, .., c_n))$. A vector $\mathbf{e} = (e_1, .., e_n) \in \prod_{i=1}^n K_i$ is a Cournot equilibrium for o if:*

$$e_i \in \arg\max_{q_i \in K_i} p\left(q_i + \sum_{j \neq i} e_j\right) q_i - c_i(q_i) \qquad \forall i = 1, .., n.$$

[9] Under our continuity assumption on f_i, if B_i is bounded so is $K_i = f_i(B_i)$.

The concept of equilibrium relies fundamentally on the notion of a dynamical system. The word "equilibrium" is not to be taken in the restrictive sense of "economic equilibrium" but in its most general sense. An equilibrium is a state[10] which does not change over time. A state is an equilibrium only in relation to a certain dynamical system and, in principle, it cannot be regarded as an equilibrium if there does not exist a criterion for determining its temporal evolution.

In Cournot's Recherches the competition of producers is formulated as a production-adjustment process converging to a unique equilibrium. Therein the idea of equilibrium is justified by the existence of a dynamical conception of competitive behavior. With no reference to a dynamical system it is not correct to consider a vector which satisfies the above definition of a Cournot equilibrium an equilibrium *stricto sensu*. What is really difficult to understand is why some analyses which pretend to provide a dynamical interpretation of Cournot equilibrium are concerned with its uniqueness. If we have good economic reasons for being interested in a particular dynamical system then we ought to study that system without expecting predetermined characteristics of the set of equilibria[11]. In particular, the multiplicity of equilibria ought not to be a cause for concern.

On the other hand we may prescind from a dynamical perspective and focus on a mere relation of causality between states. Working this way we associate with a state one consequent state that satisfies certain conditions. What we cannot do is to associate with a state two or more consequent states: this is not a causal relation since the realization of two or more states is impossible. In this work a Cournot equilibrium is regarded as the productive configuration associated with an oligopoly within the context described in the previous section. The relation between oligopoly and productive configuration is causal, hence a Cournot "equilibrium" must be thought of as a conditional prediction of a theory. In this context the term "equilibrium" is a misnomer since we are not making reference to any dynamical system; nevertheless we retain this term because of its common usage. The fact that a Cournot equilibrium is here regarded as a prediction of a theory is a first good reason for being interested in conditions ensuring the existence of a unique equilibrium: if two or more equilibria exist the realization of a Cournot equilibrium, which is unique in any case, is determined by something external to the theory, which, therefore, turns out to be incomplete.

For several years the Cournot equilibrium – because of its resemblance to the Nash equilibrium – has been thought of as the result of rational behavior and not as an arbitrary assumption on firms' behavior. Actually it was a received wisdom which has turned out to be false in several respects, even under strong assumptions on agents' knowledge. We are not going to provide any justification for the Cournot equilibrium concept since, in any case, every justification must

[10] A state of a system is a complete description of the configuration of the elements of that system. Therefore the states of a same system are mutually exclusive.

[11] Certainly the problem of uniqueness remains interesting from a merely mathematical viewpoint.

rely on assumptions which are necessarily arbitrary[12], and in this sense assuming the validity of a certain behavior is not more arbitrary than assuming the validity of the assumptions justifying that behavior.

What we want to point out now is that the importance of uniqueness is not diminished even when firms are assumed to independently forecast the realization of a Cournot equilibrium. When there exists a unique Cournot equilibrium all firms share the same forecast; this may not happen when many equilibria exist. The following example shows that a multiplicity of equilibria may generate outcomes that are inconsistent with the Cournot equilibrium concept, even when each firm independently forecasts the realization of a Cournot equilibrium.

Let $p : \mathbb{R}_+ \to \mathbb{R}_+$, $p : x \mapsto \max\{0, \min\{4 - x, 6 - 3x\}\}$ be a price function, let $K_1 := [0, 1]$ and $K_2 := [0, 1]$ be the productive capacities of firms 1 and 2, and $c_1 : [0, 1] \to \mathbb{R}_+$, $c_2 : [0, 1] \to \mathbb{R}_+$, $c_1 : x \mapsto 2x$, $c_2 : x \mapsto 2x$ be their cost functions. The set of Cournot equilibria for $(p, 2, (K_1, K_2), (c_1, c_2))$ is $E := \{(x_1, x_2) \in [1/3, 2/3] \times [1/3, 2/3] : x_1 + x_2 = 1\}$, namely the set of convex combinations of the vectors $(2/3, 1/3)$ and $(1/3, 2/3)$. Notice that for every vector in E the sum of the profits of the two firms is 1, and therefore the Pareto efficient subset of E is E. Firm 1 may forecast the realization of $(1/2, 1/2)$, thus producing $1/2$ whereas firm 2 may forecast the realization of $(2/3, 1/3)$, thus producing $1/3$. Yet $(1/2, 1/3)$ is not a Cournot equilibrium. This shows that when firms forecast the realization of a Cournot equilibrium, but not of the same Cournot equilibrium, the eventual outcome need not be a Cournot equilibrium. One may say that firms can coordinate. Actually this is a possibility we have not excluded. But in that case firms need to agree upon a Cournot equilibrium[13]. The fact that firms are able to come to an agreement is not plain: according to the previous example the set of Cournot equilibria coincides with its own Pareto efficient subset, which entails that, in E, the firms' interests are mutually conflicting.

3 - Necessary Conditions

In this section we shall establish necessary and sufficient conditions for the existence of a unique Cournot equilibrium in sets of oligopolies with linear cost functions and the same price function. Particular emphasis will be placed on the necessity of the conditions.

In the context discussed in the first section the notion of a price function is independent from the notion of a firm. This conceptual independence is

[12] Any justification for an assumption necessarily relies on other assumptions. If we try to justify these latter assumptions we still need to rely on other assumptions. This brings forth a chain of justifications which can end only with a set of unjustified assumptions.

[13] Notice that, in the previous example, the set of Cournot equilibria coincides with the set of Nash equilibria and with the set of coalition-proof Nash equilibria (in the sense of Bernheim et al. (1987)) of the game associable with the oligopoly under discussion.

preserved in our formulation of the problem of the necessity of conditions for the existence of a unique Cournot equilibrium. In many problems of entry, adoption of a technology or horizontal mergers, one cannot regard as exogenously fixed the number of firms and their cost functions. Often these problems are devised as games in extensive form in which, typically, the final subgames are games of oligopolistic competition a là Cournot. At each initial node of these final subgames the existence and the structure of an oligopoly are determined by the choices of entry, adoption of a technology or merger implemented at preceding nodes. The set of Nash equilibria for each of these final subgames coincides the set of Cournot equilibria of the oligopoly generated at that final subgame[14]. When all the oligopolies generated in a game of this type possess a unique Cournot equilibrium, we can simplify the study and the search of the pure strategy subgame perfect equilibria of the overall game[15] by replacing each final subgame with the unique tuple of Cournot equilibrium profits of the oligopoly generated at that subgame. But a proposition on the uniqueness of a Cournot equilibrium can be useful for all problems of this kind only if it considers all of the oligopolies which, in principle, can be therein generated.

The conceptual separation between the notion of a price function and the notion of a firm allows us to construct many oligopolies with the same price function but different numbers of firms, different productive capacities and different cost functions. Hence the problem of both the necessity and the sufficiency of conditions should be formulated on the set of all the oligopolies which can be formed with a same price function by allowing an indefinite number of firms to choose their cost functions from among general classes of cost functions. Several economic arguments can be posed to justify the restriction of the class of possible cost functions to some particular classes. Certainly, the reasonableness of these arguments crucially depends on the context. Given our considerations[16] on the cost functions of firms which have already incurred fixed costs, the class of convex cost functions seems to be the most significant class to be considered in such problems, hence this class will be taken into consideration in the next section when deriving sufficient conditions for the existence of a unique Cournot equilibrium. Here concern is with necessary conditions, which instead, for their being necessary, ought to be derived on a very basic class of cost functions. The case of single-product firms with increasing linear cost functions fills an important role since linear cost functions are the most basic. Establishing necessary conditions on p for the existence of a unique equilibrium for every oligopoly in O^p_{lin} entails establishing necessary conditions on p for the existence of a unique equilibrium for every oligopoly in O^p_{cvx}, in O^p_{inc} and, more generally, in every superset of O^p_{lin}. In order to avoid particular cases, which may raise objections,

[14] A plethora of models follows the above general pattern. Several examples and references can be found in the twelfth chapter of *Microeconomic Theory*, by Mas-Colell, Whinston and Green, and in any standard textbook of Industrial Organization.

[15] In particular it may simplify the investigation of conditions for the existence of a unique pure strategies subgame perfect Nash equilibrium for these types of games.

To the writer, the argument for uniqueness developed in the second section ought to be extended also to the concept of a subgame perfect Nash equilibrium.

[16] See the first section.

the problem of the necessity of conditions is formulated on the subset of O^p_{lin} whose members are oligopolies with at least two firms, namely $O^p_{lin(2)}$.

The proof of the following proposition is quite long and complex. The most important conclusion is the "only if" part of the proposition. The proof thereof is given in Appendix A. The proof of the "if" part of the proposition is in Appendix C: in that proof we show that the necessary conditions for the existence of a unique Cournot equilibrium for every element in $O^p_{lin(2)}$ suffice for the existence of a unique Cournot equilibrium for every element in O^p_{cvx}, which is a superset of $O^p_{lin(2)}$.

Proposition 5 *Let $p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a price function, continuous over $\mathbb{R}_{++}$, null at some point of $\mathbb{R}_{++}$, such that $p(0) = \limsup_{x\rightarrow 0^+} p(x)$ and $\liminf_{x\rightarrow 0^+} p(x) > 0$. Let $q_0 := \min\{x \in \mathbb{R}_{++} : p(x) = 0\}$ and $q_\mu : [0, q_0) \rightarrow \mathbb{R}$, $q_\mu : z \mapsto \min \arg\max_{x\in[0,q_0]} p(z + x)\, x$. Every oligopoly in $O^p_{lin(2)}$ has a unique Cournot equilibrium if and only if:*

1. p is continuously differentiable over $(0, q_0)$,
2. $p(q + z)\, q$ is quasiconcave in q over $\mathbb{R}_+$, $\forall z \in \mathbb{R}_+$,
3. $p(q + z)\, q$ is strictly concave in q over $(0, q_\mu(z))$, $\forall z \in [0, q_0)$,
4. for every integer $m \geq 1$, $\forall x \in (0, q_0)$,$\forall y \in [0, x]$,$\forall z \in (0, q_0 - x)$, $p'(x)\, y + mp(x) > 0$ entails that $p'(x)\, y + mp(x) > p'(x + z)(y + z) + mp(x + z)$.

Let us now discuss the main steps of the proof of the "only if" part (Appendix A). We first show that p must be decreasing over $(0, q_0)$ and nonincreasing over $\mathbb{R}_+$[17]. The reader may be surprised not to find this condition enumerated in the list of necessary conditions, yet, since we are dealing with necessary conditions, its absence does not affect the necessity of the other four conditions. We shall return to this point later when dealing with the "if" part of the proposition.

Having established that p must be decreasing over $(0, q_0)$ and nonincreasing over $\mathbb{R}_+$, we restrict to such a set of functions to investigate the necessity of other properties. The reason for this restriction is simple: if the property does not hold then we can construct an oligopoly in $O^p_{lin(2)}$ with at least two Cournot equilibria. This way of proving the necessity of a condition is common to all our proofs of necessary conditions; it is important to remark that the structure of the proofs is constructive in the sense that we show how to construct an oligopoly with at least two Cournot equilibria whenever the condition under consideration does not hold. It must be noticed that the monotonicity of p, which is usually assumed without clear justification, is here proved to be necessary.

Then we show that $p(q + z)\, q$ must be strictly concave in q over $(0, q_\mu(z))$, $\forall z \in [0, q_0)$. The proof of this condition is split into two parts. First we show the

[17] Notice that when p is nonincreasing over $\mathbb{R}_+$ then p must be continuous over $\mathbb{R}_+$ (recall that p is continuous over $\mathbb{R}_{++}$ and $p(0) = \limsup_{x\rightarrow 0^+} p(x)$) moreover $p(x) > 0$ $\forall x \in [0, q_0)$ and $p(x) = 0$ $\forall x \geq q_0$.

necessity of concavity and then the necessity of strict concavity; this separation simplifies the proof. This condition entails that the revenue of a firm when the overall production of other firms is fixed at any $z \in [0, q_0)$, must be strictly concave over $(0, q_\mu(z))$, where $q_\mu(z)$ is the least maximizer of the revenue of that firm when the overall production of the firms is z.

Continuing to restrict the set of admissible functions we find that the continuous differentiability of p over $(0, q_0)$ is a necessary condition: this condition entails that, over $(0, q_0)$, the right-hand and the left-hand marginal revenue must exist and be equal, and that, over $(0, q_0)$, the marginal revenue must vary continuously. This condition is an usual assumption in the literature on the existence of a Cournot equilibrium, yet its necessity has never been proven. This result is obtained by means of linear cost functions, which, certainly, possess a linear and differentiable extension to $\mathbb{R}$. Therefore the necessity of the continuous differentiability of p cannot be supposed to be determined by the nondifferentiability of some cost function at some interior point of its domain.

Another condition usually assumed in the literature on the existence of a Cournot equilibrium appears as a necessary condition: $p(q+z)q$ must be quasiconcave[18] in q over $\mathbb{R}_+$, $\forall z \in \mathbb{R}_+$. This condition entails that the revenue of a firm, when the aggregate production of other firms is fixed at any nonnegative level z, must be quasiconcave. In the literature this assumption is usually introduced to ensure the existence of a Cournot equilibrium. It is noteworthy that we do not show that the absence of this condition entails the emptiness of the set of equilibria, but, quite the reverse, the existence of a multiplicity of equilibria.

The last condition is more complex. It says that, given an overall production $x \in (0, q_0)$, if, for a positive integer m, the marginal revenue of a firm that produces $y/m \leq x/m$ is positive, then it must be strictly greater than the marginal revenue of the same firm when the overall production is $x + z$ and its production is $(y+z)/m$, for every $z \in (0, q_0 - x)$. Another interpretation of the same condition is the following: whenever for an overall production $x \in (0, q_0)$ there exist two sets of firms, the first with $m \geq 1$ firms and the second with $l \geq 0$ firms, such that the overall production of the m firms is $y \leq x$, if the value of the sum of their marginal revenues is positive then it must be strictly greater than the value of the sum of their marginal revenues when their overall production increases by z, for every $z \in (0, q_0 - x)$, and the production of the other l firms does not change.

We must notice that when the first three conditions hold the fourth condition effectively restricts the set of admissible functions only if $m > 1$; since we shall not use this implication we do not prove it; yet it can be easily verified by the reader.

In the following section we shall prove that conditions 1 to 4 suffice for the existence of a unique Cournot equilibrium for every oligopoly in O^p_{cvx}. Since

[18] Let A be a real interval. A function $g : A \rightarrow \mathbb{R}$ is quasiconcave if $\forall x_1, x_2 \in A, \forall l \in \mathbb{R}$, $\forall \alpha \in [0,1]$ $g(x_1) \geq l$ and $g(x_2) \geq l$ implies that $g(\alpha x_1 + (1-\alpha) x_2) \geq l$; g is strictly quasiconcave if $\forall x_1, x_2 \in A, \forall l \in \mathbb{R}$, $\forall \alpha \in (0,1)$ $g(x_1) \geq l$, $g(x_2) \geq l$ and $x_1 \neq x_2$ implies that $g(\alpha x_1 + (1-\alpha) x_2) > l$.

$O^p_{lin(2)} \subset O^p_{cvx}$ we can easily deduce that conditions 1 to 4 suffice for the existence of a unique Cournot equilibrium for every oligopoly in $O^p_{lin(2)}$. Thus the validity of the "if" part of previous proposition should be clear. We leave to the next section the detailed proof of this, here we hark back to an issue we have hinted at before. A necessary condition requires p to be decreasing over $(0, q_0)$ and nonincreasing over $\mathbb{R}_+$, we have not listed this condition with the other necessary conditions. This lack not only does not affect the necessity of the other four conditions but does not affect the set of admissible functions either. The reason is simple: when conditions 1 through 3 hold then p must be decreasing over $(0, q_0)$ and nonincreasing over $\mathbb{R}_+$. The following proposition gives a formal statement of this; we also notice that conditions 1 through 3 imply that the derivative of p must be negative over $(0, q_0)$.[19]

Proposition 6 *Let $p : \mathbb{R}_+ \to \mathbb{R}_+$ be a price function, continuous over $\mathbb{R}_{++}$, null at some point of $\mathbb{R}_{++}$, such that $p(0) = \limsup_{x\to 0^+} p(x)$ and $\liminf_{x\to 0^+} p(x) > 0$. Let $q_0 := \min\{x \in \mathbb{R}_{++} : p(x) = 0\}$ and $q_\mu : [0, q_0) \to \mathbb{R}$, $q_\mu : z \mapsto \min \arg\max_{x\in[0,q_0]} p(z + x)\, x$. p is decreasing over $[0, q_0]$, nonincreasing over $\mathbb{R}_+$, and in particular $p'(x) < 0$ $\forall x \in (0, q_0)$ and $p(x) = 0$ $\forall x \geq q_0$, if:*

1. *p is continuously differentiable over $(0, q_0)$,*
2. *$p(q + z)\, q$ is strictly concave in q over $(0, q_\mu(z))$, $\forall z \in [0, q_0)$,*
3. *$p(q + z)\, q$ is quasiconcave in q over $\mathbb{R}_+$, $\forall z \in \mathbb{R}_+$.*

The simple proof of this proposition is in Appendix B.

Appendix A

Proposition 7 *Let $p : \mathbb{R}_+ \to \mathbb{R}_+$ be a price function, continuous over $\mathbb{R}_{++}$, null at some point of $\mathbb{R}_{++}$, such that $p(0) = \limsup_{x\to 0^+} p(x)$ and $\liminf_{x\to 0^+} p(x) > 0$. Let $q_0 := \min\{x \in \mathbb{R}_{++} : p(x) = 0\}$. Every oligopoly in $O^p_{lin(2)}$ has a unique Cournot equilibrium only if p is nonincreasing over $\mathbb{R}_+$ and decreasing over $(0, q_0)$.*

Proof. To prove the proposition it suffices to prove that for every $z > 0$, such that $p(z) > 0$, p must be decreasing over $(0, z)$. Let us admit that there exists a value $z > 0$, such that $p(z) > 0$ and p is not decreasing over $(0, z)$. Therefore, by continuity of p, there exist two values, say x° and $x^{\circ\circ}$, such that $x^{\circ\circ} > x^\circ > 0$, $p(x^{\circ\circ}) \geq p(x^\circ)$ and $p(x) > 0$ $\forall x \in [x^\circ, x^{\circ\circ}]$. If $p(x^{\circ\circ}) = \max_{x\in[x^\circ,x^{\circ\circ}]} p(x)$ let $x_2 := x^{\circ\circ}$, otherwise let $x_2 := \min \arg\max_{x\in(x^\circ,x^{\circ\circ}]} p(x)$,

[19] Let $A \subseteq \mathbb{R}$ be open. If $f : A \to \mathbb{R}$ is differentiable and decreasing then f' may be null.

and let $x_1 := \min \arg\min_{x\in[x^\circ,x_2]} p(x)$. Clearly $x_2 > x_1 > 0$, $p(x_2) \geq p(x) > 0$ $\forall x \in [x_1, x_2]$ and $0 < p(x_1) \leq p(x)$ $\forall x \in [x_1, x_2]$. If p is constant over $[x_1, x_2]$ let $\bar{x} := x_1 + \frac{x_2-x_1}{2}$ otherwise let $\bar{x} := \min\{ x \in (x_1, x_2) : p(x) = \frac{p(x_2)+p(x_1)}{2} \}$. In either case

$$x_2 > \bar{x} > x_1 > 0,$$

$$p(x_2) \geq p(x) > 0 \ \forall x \in [x_1, x_2], \tag{1}$$

and

$$p(\bar{x}) \geq p(x) > 0 \ \forall x \in [x_1, \bar{x}]. \tag{2}$$

Let $\alpha := p(x_2)$ and $\beta := \frac{p(x_1)}{2} > 0$. Let $\nu \geq 1$ be an integer number such that $\nu y = \bar{x}$ for some $y \in (0, \bar{x} - x_1)$. By (1) we have that:

$$p(x_2)(x_2 - \bar{x}) - \alpha(x_2 - \bar{x}) \geq p(\nu y + q) q - \alpha q \ \forall q \in [0, x_2 - \bar{x}], \tag{3}$$

$$p(\nu y + 0) 0 - \alpha 0 \geq p(\nu y + q) q - \alpha q \ \forall q \in [0, x_2 - \bar{x}], \tag{4}$$

and

$$p(x_2) y - \beta y \geq p(x_2 - y + q) q - \beta q \ \forall q \in [0, y]. \tag{5}$$

By (2) we have that:

$$p(\nu y) y - \beta y \geq p((\nu - 1) y + q) q - \beta q \ \forall q \in [0, y]. \tag{6}$$

Let us now consider the oligopoly $o := (p, \nu + 1, (K_1, ..., K_{\nu+1}), (c_1, ..., c_{\nu+1}))$ where $K_i := [0, y]$, $c_i : K_i \to \mathbb{R}_+$, $c_i : x \mapsto \beta x$, $\forall i = 1, .., \nu$, and $K_{\nu+1} := [0, x_2 - \bar{x}]$, $c_{\nu+1} : K_{\nu+1} \to \mathbb{R}_+$, $c_{\nu+1} : x \mapsto \alpha x$. It is trivial that $o \in O^p_{lin(2)}$. We can show that $\mathbf{e}_1 := (y, ..., y, x_2 - \bar{x})$ and $\mathbf{e}_2 := (y, .., y, 0)$ are two different Cournot equilibria for $(p, \nu + 1, (K_1, ..., K_{\nu+1}), (c_1, .., c_{\nu+1}))$. Since $x_2 > \bar{x}$ then $\mathbf{e}_1 \neq \mathbf{e}_2$.

Let us consider $\mathbf{e}_1$. The aggregate production of the first ν firms is νy, by (3) we know that $(x_2 - \bar{x})$ maximizes the profit function of firm $\nu + 1$ when the other firms produce νy. For any firm $i \neq \nu + 1$, the aggregate production of the other firms is $(\nu - 1) y + (x_2 - \bar{x}) = x_2 - y$, by (5) we know that y maximizes the profit function of firm i when the other firms produce $x_2 - y$. Therefore $\mathbf{e}_1$ is a Cournot equilibrium for o.

Let us consider $\mathbf{e}_2$. The aggregate production of the first ν firms is νy, by (4) we know that 0 maximizes the profit function of firm $\nu + 1$ when the other firms produce νy. For any firm $i \neq \nu + 1$: the aggregate production of the other firms is $(\nu - 1) y$, by (6) we know that y maximizes the profit function of firm i when the other firms produce $(\nu - 1) y$. Therefore $\mathbf{e}_2$ is a Cournot equilibrium for o.

Hence whenever p is not nonincreasing over $\mathbb{R}_+$ or not decreasing over $(0, q_0)$, we can construct an oligopoly in $O^p_{lin(2)}$, with two different equilibria. ■

Proposition 8 *Let $p : \mathbb{R}_+ \to \mathbb{R}_+$ be a price function, continuous over $\mathbb{R}_{++}$, null at some point of $\mathbb{R}_{++}$, such that $p(0) = \limsup_{x\to 0^+} p(x)$ and $\liminf_{x\to 0^+} p(x) >$*

0. *Let* $q_0 := \min\{x \in \mathbb{R}_{++} : p(x) = 0\}$ *and* $q_\mu : [0, q_0) \to \mathbb{R}$, $q_\mu : z \mapsto \min \arg\max_{x\in[0,q_0]} p(z+x)\,x$. *Every oligopoly in* $O^p_{lin(2)}$ *has a unique Cournot equilibrium only if* $p(q+z)\,q$ *is concave in* q *over* $(0, q_\mu(z))$, $\forall z \in [0, q_0)$.

Proof. By the previous proposition p must be nonincreasing over $\mathbb{R}_+$ and decreasing over $(0, q_0)$. Let us admit that there exists a value $\tilde{y} \in [0, q_0)$ such that $p(q+\tilde{y})\,q$ is not concave in q over $(0, q_\mu(\tilde{y})) \subset [0, q_0]$. This entails that there exist four real values $\tilde{y} \in [0, q_0)$, x°, $x^{\circ\circ}$, and $\gamma \in (0,1)$ such that $q_\mu(\tilde{y}) > x^{\circ\circ} > x^\circ > 0$, and

$$\gamma p(x^\circ + \tilde{y})\,x^\circ + (1-\gamma)\,p(x^{\circ\circ} + \tilde{y})\,x^{\circ\circ} > p(\bar{x} + \tilde{y})\,\bar{x}, \tag{1}$$

where $\bar{x} := \gamma x^\circ + (1-\gamma)\,x^{\circ\circ}$. By continuity of p there exists a positive value $\bar{y} < q_0 - x^{\circ\circ}$ such that for every $z \in [0, \bar{y}]$

$$\gamma p(x^\circ + \tilde{y} + z)\,x^\circ + (1-\gamma)\,p(x^{\circ\circ} + \tilde{y} + z)\,x^{\circ\circ} > p(\bar{x} + \tilde{y} + z)\,\bar{x}. \tag{2}$$

Since $q_\mu(\tilde{y}) > x^{\circ\circ}$, by continuity of p there exists a positive value $\breve{y} < \bar{y}$ such that for every $z \in [0, \breve{y}]$, $q_\mu(\tilde{y} + z) > x^{\circ\circ}$. Let

$$y' := \min\{x \in (0, \breve{y}] : p(\tilde{y} + x)\,x = p(\breve{y} + \tilde{y})\,\breve{y}\}.$$

Let $y := \tilde{y} + y'$ and $q^y_\mu := q_\mu(y)$. By construction $p(y)\,y' \geq p(y+z)\,(y'+z)$ $\forall z \in [-y', 0]$; therefore

$$-\frac{p(y+z)}{y'} \leq \frac{p(y+z) - p(y)}{z} \quad \forall z \in [-y', 0). \tag{3}$$

Let $\lambda_1 := \min_{z\in[-y',0]} \frac{-p(y+z)}{y'} < 0$, we have that

$$\lambda_1 \leq \frac{p(y+z) - p(y)}{z} \quad \forall z \in [-y', 0). \tag{4}$$

Let $m_1 : \mathbb{R}_+ \to \mathbb{R}$, $m_2 : \mathbb{R}_+ \to \mathbb{R}$, $m_3 : \mathbb{R}_+ \to \mathbb{R}$ be three functions such that $m_1 : \alpha \mapsto \max_{x\in[0,\bar{x}]} [p(y+x)\,x - \alpha x]$, $m_2 : \alpha \mapsto \max_{x\in[\bar{x},q^y_\mu]} [p(y+x)\,x - \alpha x]$, and $m_3 : \alpha \mapsto \max_{x\in[0,q^y_\mu]} [p(y+x)\,x - \alpha x]$. Since by definition $p\left(q^y_\mu + y\right) q^y_\mu > p(q+y)\,q$ $\forall q < q^y_\mu$ then $m_2(0) > m_1(0)$. Let $\alpha^\circ := 2\frac{p\left(q^y_\mu + y\right) q^y_\mu}{\bar{x}}$. Since $p(y+0)\,0 - \alpha^\circ 0 = 0$ then $m_1(\alpha^\circ) \geq 0$; since $0 > p\left(q^y_\mu + y\right) q^y_\mu - \alpha^\circ \bar{x} > \max_{x\in[\bar{x},q^y_\mu]} [p(y+x)\,x - \alpha^\circ x]$ then $m_2(\alpha^\circ) < m_1(\alpha^\circ)$. Since m_1,m_2 and m_3 are continuous over $[0, \alpha^\circ]$ there exists a value say $\bar{\alpha}$ such that $\bar{\alpha} > 0$ and $m_2(\bar{\alpha}) = m_1(\bar{\alpha})$. By (2) we have that

$$\gamma\,[p(x^\circ + y) - \bar{\alpha}]\,x^\circ + (1-\gamma)\,[p(x^{\circ\circ} + y) - \bar{\alpha}]\,x^{\circ\circ} > [p(\bar{x} + y) - \bar{\alpha}]\,\bar{x},$$

this entails that $[p(x^\circ + y) - \bar{\alpha}]\,x^\circ > [p(\bar{x} + y) - \bar{\alpha}]\,\bar{x}$ or $[p(x^{\circ\circ} + y) - \bar{\alpha}]\,x^{\circ\circ} > [p(\bar{x} + y) - \bar{\alpha}]\,\bar{x}$ or both, therefore $\bar{x} \notin \arg\max_{x\in[0,q^y_\mu]} [p(y+x)\,x - \bar{\alpha}x]$. Since $\bar{x} \notin$

$\underset{x\in[0,q_\mu^y]}{\arg\max}\,[p\,(y+x)\,x-\bar{\alpha}x]$ and $m_2\,(\bar{\alpha})=m_1\,(\bar{\alpha})=m_3\,(\bar{\alpha})$ there must exist two values, say x_1 and x_2 such that:

$$q_\mu^y \geq x_2 > \bar{x} > x_1 \geq 0,$$

$$x_1 := \min \underset{q\in[0,q_\mu^y]}{\arg\max}\,[p\,(y+q)\,q-\bar{\alpha}q]\,, \tag{5}$$

$$x_2 := \max \underset{x\in[0,q_\mu^y]}{\arg\max}\,[p\,(y+q)\,q-\bar{\alpha}q]\,. \tag{6}$$

Let $\lambda_2 := \min\limits_{q\in[-x_2,0]} \frac{-p(y+x_2+q)+\bar{\alpha}}{x_2}$. By (6) we have that $p\,(y+x_2)\,x_2-\bar{\alpha}x_2 \geq p\,(y+x_2+q)\,(x_2+q)-\bar{\alpha}\,(x_2+q)$ $\forall q \in [-x_2,0]$, therefore $\frac{-p(y+x_2+q)+\bar{\alpha}}{x_2} \leq \frac{p(y+x_2+q)-p(y+x_2)}{q} \forall q \in [-x_2,0)$ and

$$\lambda_2 \leq \frac{p\,(y+x_2+q)-p\,(y+x_2)}{q} \forall q \in [-x_2,0)\,.$$

If $x_1>0$ let $\lambda_3 := \min\limits_{q\in[-x_1,0]} \frac{-p(y+x_1+q)+\bar{\alpha}}{x_1} < 0$, in this case by (5) we have that $p\,(y+x_1)\,x_1-\bar{\alpha}x_1 \geq p\,(y+x_1+q)\,(x_1+q)-\bar{\alpha}\,(x_1+q)$ $\forall q \in [-x_1,0)$, therefore $\frac{-p(y+x_1+q)+\bar{\alpha}}{x_1} \leq \frac{p(y+x_1+q)-p(y+x_1)}{q} \forall q \in [-x_1,0)$ and

$$\lambda_3 \leq \frac{p\,(y+x_1+q)-p\,(y+x_1)}{q} \forall q \in [-x_1,0)\,.$$

If $x_1>0$ let $\lambda := -\min\{\lambda_2,\lambda_3\}>0$ and $\frac{y}{\nu} \in (0,\min\{1,x_1\})$ for some integer $\nu \geq 1$, if $x_1=0$ let $\lambda := -\min\{\lambda_1,\lambda_2\}>0$ and let $\frac{y}{\nu} \in (0,\min\{1,x_2,y'\})$ for some integer $\nu \geq 1$. In either case we have that:

$$-\lambda \leq \frac{p\,(y+x_1+q)-p\,(y+x_1)}{q} \forall q \in \left[-\frac{y}{\nu},0\right)\,, \tag{7}$$

$$-\lambda \leq \frac{p\,(y+x_2+q)-p\,(y+x_2)}{q} \forall q \in \left[-\frac{y}{\nu},0\right)\,. \tag{8}$$

By (7) we have that $-\lambda q \geq p\,(y+x_1+q)-p\,(y+x_1) \forall q \in \left[-\frac{y}{\nu},0\right)$, since $0<\frac{y}{\nu}<1$ then $-\lambda q \geq p\,(y+x_1+q)\,\frac{y}{\nu}-p\,(y+x_1)\,\frac{y}{\nu}$ $\forall q \in \left[-\frac{y}{\nu},0\right)$, hence $-\lambda\left(q+\frac{y}{\nu}-\frac{y}{\nu}\right) \geq p\,(y+x_1+q)\left(\frac{y}{\nu}+q\right)-p\,(y+x_1)\,\frac{y}{\nu}$ $\forall q \in \left[-\frac{y}{\nu},0\right)$, therefore

$$p\,(y+x_1)\,\frac{y}{\nu}-\lambda\frac{y}{\nu} \geq p\,(y+x_1+q)\left(\frac{y}{\nu}+q\right)-\lambda\left(\frac{y}{\nu}-q\right) \;\; \forall q \in \left[-\frac{y}{\nu},0\right)\,. \tag{9}$$

Similarly, by (8) we can obtain:

$$p\,(y+x_2)\,\frac{y}{\nu}-\lambda\frac{y}{\nu} \geq p\,(y+x_2+q)\left(\frac{y}{\nu}+q\right)-\lambda\left(\frac{y}{\nu}-q\right) \;\; \forall q \in \left[-\frac{y}{\nu},0\right)\,. \tag{10}$$

Let us consider the oligopoly $o := (p,\nu+1,(K_1,..,K_{\nu+1})\,,(c_1,..,c_{\nu+1}))$ where $K_i=[0,y/\nu]$, $c_i : K_i \to \mathbb{R}_+$, $c_i : x \mapsto \lambda x$ for $i=1,..,\nu$, $K_{\nu+1}=\left[0,q_\mu^y\right]$,

$c_{\nu+1} : K_{\nu+1} \to \mathbb{R}_+$, $c_{\nu+1} : x \mapsto \bar{\alpha}x$. It is trivial that $o \in O^p_{lin(2)}$. Let $\mathbf{e}_1 := \left(\frac{y}{\nu}, .., \frac{y}{\nu}, x_1\right)$ and $\mathbf{e}_2 := \left(\frac{y}{\nu}, .., \frac{y}{\nu}, x_2\right)$.

We can easily verify that $\mathbf{e}_1$ and $\mathbf{e}_2$ are Cournot equilibria for o. Let us consider $\mathbf{e}_1$. The aggregate production of the first ν firms is y and, by (5), x_1 maximizes the profit function of firm $\nu+1$ when the aggregate production of the other firms is y. For any firm $i \neq \nu+1$ the aggregate production of the other firms is $\frac{\nu-1}{\nu}y + x_1$, by (9), $\frac{y}{\nu}$ maximizes the profit function of firm i when the aggregate production of the other firms is $\frac{\nu-1}{\nu}y + x_1$. Therefore $\mathbf{e}_1$ is a Cournot equilibrium. Considering (6) and (10) it can be similarly verified that $\mathbf{e}_2$ $(\neq \mathbf{e}_1)$ is a Cournot equilibrium.

Hence whenever $p(q+z)q$ is not concave in q over $(0, q_\mu(z))$ for some $z \in [0, q_0)$ we can construct an oligopoly in $O^p_{lin(2)}$ with two different equilibria. ■

Proposition 9 *Let* $p : \mathbb{R}_+ \to \mathbb{R}_+$ *be a price function, continuous over* $\mathbb{R}_{++}$*, null at some point of* $\mathbb{R}_{++}$*, such that* $p(0) = \limsup_{x \to 0^+} p(x)$ *and* $\liminf_{x \to 0^+} p(x) > 0$*. Let* $q_0 := \min\{x \in \mathbb{R}_{++} : p(x) = 0\}$ *and* $q_\mu : [0, q_0) \to \mathbb{R}$*,* $q_\mu : z \mapsto \min \arg\max_{x \in [0, q_0]} p(z+x)x$*. Every oligopoly in* $O^p_{lin(2)}$ *has a unique Cournot equilibrium only if* $p(q+z)q$ *is strictly concave in* q *over* $(0, q_\mu(z))$*,* $\forall z \in [0, q_0)$*.*

Proof. By the previous propositions we know that $p(q+z)q$ must be concave in q over $(0, q_\mu(z))$, $\forall z \in [0, q_0)$ and p must be decreasing over $(0, q_0)$. Let us admit that there exists a value $y \in [0, q_0)$ such that $p(q+y)q$ is not strictly concave in q over $(0, q_\mu(y))$. This entails that there exist three real values $y \in [0, q_0)$, $x^{\circ\circ}$ and x° such that $q_\mu(y) > x^{\circ\circ} > x^\circ > 0$ and $\forall \gamma \in [0, 1]$:

$$\begin{aligned} & \gamma p(x^\circ + y)x^\circ + (1-\gamma)p(x^{\circ\circ} + y)x^{\circ\circ} = \\ = \; & p(\gamma x^\circ + (1-\gamma)x^{\circ\circ} + y)(\gamma x^\circ + (1-\gamma)x^{\circ\circ}). \end{aligned} \tag{1}$$

Let $\alpha := \frac{p(x^{\circ\circ}+y)x^{\circ\circ} - p(x^\circ+y)x^\circ}{x^{\circ\circ}-x^\circ} > 0$. Let $x_1 := \frac{2}{3}x^\circ + \frac{1}{3}x^{\circ\circ}$ and $x_2 := \frac{1}{3}x^\circ + \frac{2}{3}x^{\circ\circ}$. The reader can easily verify that $q_\mu(y) > x_2 > x_1 > 0$ and that $\forall q \in [0, q_\mu(y)]$:

$$p(y+x_1)x_1 - \alpha x_1 \geq p(y+q)q - \alpha q, \tag{2}$$

$$p(y+x_2)x_2 - \alpha x_2 \geq p(y+q)q - \alpha q. \tag{3}$$

If $y = 0$ consider the oligopoly $\bar{o} := (p, 2, (K_1, K_2), (c_1, c_2))$, where $K_1 = K_2 := [0, x_2]$, $c_1 : K_1 \to \mathbb{R}_+$, $c_2 : K_2 \to \mathbb{R}_+$, $c_1 : x \mapsto ax$, $c_2 : x \mapsto p(0)x$: it can be easily verified that $(x_1, 0)$ and $(x_2, 0)$ are two Cournot equilibria for $\bar{o}$. Hence $y > 0$. Let $z_1 \in (x^\circ, x_1)$ and $z_2 \in (x_2, x^{\circ\circ})$. Since p is continuously differentiable over $(x^\circ, x^{\circ\circ})$ then $\max_{x \in [z_1, z_2]} -p'(x)$ exists, let $\tau > 0$ be a real number such that $\tau > \max_{x \in [z_1, z_2]} -p'(x)$. Let $\nu \geq 1$ be an integer such that $\frac{y}{\nu} < x_1 - x^\circ$ and $\frac{y}{\nu} < \frac{p(x_2+y)}{\tau}$. Let $\beta > 0$ be a real such that $\beta < -\tau\frac{y}{\nu} + p(x_2+y)$. Since $\beta < -\tau\frac{y}{\nu} + p(q + x_2 + y)$ then $\forall q \in \left[0, \frac{y}{\nu}\right]$:

$$-\tau q + p\left(q + x_1 + y\frac{\nu-1}{\nu}\right) - \beta > 0,$$

$$-\tau q+p\left(q+x_2+y\frac{\nu-1}{\nu}\right)-\beta>0.$$

Therefore $\forall q\in\left[0,\frac{y}{\nu}\right]$:

$$p'\left(q+x_1+y\frac{\nu-1}{\nu}\right)q+p\left(q+x_1+y\frac{\nu-1}{\nu}\right)-\beta>0,$$

$$p'\left(q+x_2+y\frac{\nu-1}{\nu}\right)q+p\left(q+x_2+y\frac{\nu-1}{\nu}\right)-\beta>0.$$

This entails that $p\left(x_1+y\frac{\nu-1}{\nu}+q\right)q-\beta q$ and $p\left(x_2+y\frac{\nu-1}{\nu}+q\right)q-\beta q$ are nondecreasing in q over $\left[0,\frac{y}{\nu}\right]$.

Therefore $\forall q\in\left[0,\frac{y}{\nu}\right]$:

$$p\left(y+x_1\right)\frac{y}{\nu}-\beta\frac{y}{\nu}\geq p\left(x_1+y\frac{\nu-1}{\nu}+q\right)q-\beta q, \tag{4}$$

$$p\left(y+x_2\right)\frac{y}{\nu}-\beta\frac{y}{\nu}\geq p\left(x_2+y\frac{\nu-1}{\nu}+q\right)q-\beta q. \tag{5}$$

Now let us consider the oligopoly $o:=\left(p,\nu+1,\left(K_1,..,K_{\nu+1}\right),\left(c_1,..,c_{\nu+1}\right)\right)$ where $K_i=[0,y/\nu]$ for $i=1,..,\nu$, $K_{\nu+1}=[0,q_\mu(y)]$, $c_i:K_i\rightarrow\mathbb{R}_+$ for $i=1,...,\nu$, $c_i:x\mapsto\beta x$ for $i=1,..,\nu$, and $c_{\nu+1}:x\mapsto\alpha x$. Let $\mathbf{e}_1:=\left(\frac{y}{\nu},..,\frac{y}{\nu},x_1\right)$ and $\mathbf{e}_2:=\left(\frac{y}{\nu},..,\frac{y}{\nu},x_2\right)$, it is easy to verify that $\mathbf{e}_1$ and $\mathbf{e}_2$ are Cournot equilibria of $\left(p,\nu+1,\left(K_1,..,K_{\nu+1}\right),\left(c_1,..,c_{\nu+1}\right)\right)$. It is trivial that $o\in O^p_{lin(2)}$.

Let us consider $\mathbf{e}_1$: each of the first ν firms produces $\frac{y}{\nu}$ and firm $\nu+1$ produces x_1. Let us consider firm $\nu+1$: in $\mathbf{e}_1$ the aggregate production of the other firms is y, by (2) we know that in this case x_1 maximizes the profit function of firm $\nu+1$. Let us consider any firm i such that $i\neq\nu+1$: in $\mathbf{e}_1$ the aggregate production of the other firms is $(\nu-1)\frac{y}{\nu}+x_1$, by (4) we know that in this case $\frac{y}{\nu}$ maximizes the profit function of firm i. Therefore $\mathbf{e}_1$ is a Cournot equilibrium. Considering (3) and (5) it can be similarly shown that $\mathbf{e}_2$ is a Cournot equilibrium. Since $x_2\neq x_1$ the equilibria do not coincide.

Hence whenever $p(q+z)q$ is not strictly concave in q over $(0,q_\mu(z))$, for some $z\in[0,q_0)$, we can construct an oligopoly in $O^p_{lin(2)}$ with two different equilibria. ■

Proposition 10 *Let $p:\mathbb{R}_+\rightarrow\mathbb{R}_+$ be a price function, continuous over $\mathbb{R}_{++}$, null at some point of $\mathbb{R}_{++}$, such that $p(0)=\limsup_{x\rightarrow0^+}p(x)$ and $\liminf_{x\rightarrow0^+}p(x)>0$. Let $q_0:=\min\{x\in\mathbb{R}_{++}:p(x)=0\}$. Every oligopoly in $O^p_{lin(2)}$ has a unique Cournot equilibrium only if p is continuously differentiable over $(0,q_0)$.*

Proof. Let $q_\mu:[0,q_0)\rightarrow\mathbb{R}$, $q_\mu:z\mapsto\min\arg\max_{x\in[0,q_0]}p(z+x)x$. By the previous propositions p must be nonincreasing over $\mathbb{R}_+$, decreasing over $(0,q_0)$ and $p(q+z)q$ must be strictly concave in q over $(0,q_\mu(z))$, $\forall z\in[0,q_0)$. The proof consists of two parts. Firstly we prove that the right-hand and the left-hand derivative of p must exist at every point in $(0, q_0)$, secondly we show that they must coincide and therefore that the differentiability of p is a necessary condition at every point in $(0, q_0)$. The continuity of p' is easily deduced by the one-sided continuity properties of the right-hand and left-hand derivatives of p.

Now we show that at every value $\hat{x} \in (0, q_0)$ the right-hand and the left-hand derivative of p, respectively denoted by p'_+ and p'_-, must exist, and that there exists an open interval $I_{\hat{x}}$ of $\hat{x}$ such that p is Lipschitz continuous over $I_{\hat{x}}$ and $p'_+ (\hat{x}) \leq p'_- (\hat{x})$.

Let us consider any value $\hat{x} \in (0, q_0)$. We firstly prove that there exists a value $\bar{\varepsilon}$ such that $q_\mu (\hat{x} - \varepsilon) > \varepsilon$, $\forall \varepsilon \in [0, \bar{\varepsilon}]$. Since (by definition) $p (\hat{x} + q_\mu (\hat{x})) \, q_\mu (\hat{x}) > p (\hat{x} + z) \, z \; \forall z \in [0, q_\mu (\hat{x}))$ and since $p (\hat{x} + z) \, z$ is strictly concave in z over $(0, q_\mu (\hat{x}))$ and continuous in z over $[0, q_\mu (\hat{x})]$ then $p (\hat{x} + z) \, z$ is increasing in z over $[0, q_\mu (\hat{x})]$. Hence $p (\hat{x} + q_\mu (\hat{x})) \, q_\mu (\hat{x}) > p (\hat{x} + z) \, z + k$ $\forall z \leq q_\mu (\hat{x}) / 2$ for some $k > 0$. By continuity of p there must exist a value $\bar{\varepsilon} \in \left(0, \min \left\{ \frac{q_\mu(\hat{x})}{2}, \hat{x} \right\}\right)$ such that

$$p (\hat{x} - \bar{\varepsilon} + q_\mu (\hat{x})) \, q_\mu (\hat{x}) > p (\hat{x} - \bar{\varepsilon} + z) \, z \quad \forall z \leq \frac{q_\mu (\hat{x})}{2}.$$

Since $p (\hat{x} + z - \bar{\varepsilon}) \, z$ is increasing in z over $[0, q_\mu (\hat{x} - \bar{\varepsilon})]$, then we have that

$$\frac{q_\mu (\hat{x})}{2} \leq q_\mu (\hat{x} - \bar{\varepsilon}) \, .$$

Since $q_\mu (\hat{x} - \bar{\varepsilon}) \geq \frac{q_\mu(\hat{x})}{2}$ and $\frac{q_\mu(\hat{x})}{2} > \bar{\varepsilon}$, then $q_\mu (\hat{x} - \bar{\varepsilon}) > \bar{\varepsilon}$. Since $q_\mu (\hat{x} - \bar{\varepsilon}) > \bar{\varepsilon} > 0$ we can choose a compact real interval $A \subset (0, q_\mu (\hat{x} - \bar{\varepsilon}))$, with nonempty interior $\mathring{A}$, such that $\bar{\varepsilon} \in \mathring{A}$. Since $p (\hat{x} + z - \bar{\varepsilon}) \, z$ is strictly concave in z over $(0, q_\mu (\hat{x} - \bar{\varepsilon}))$ then $p (\hat{x} + z - \bar{\varepsilon}) \, z$ is strictly concave in z over A. We know that since A is a compact subset of the open set $(0, q_\mu (\hat{x} - \bar{\varepsilon}))$, then $p (\hat{x} + z - \varepsilon) \, z$ is Lipschitz continuous over A and its left-hand and right-hand derivative exist at every point in A, therefore p'_+ and p'_- are respectively right-continuous and left-continuous at $\hat{x}$ and $p'_+ (\hat{x}) \leq p'_- (\hat{x})$. Thus we can conclude that at every $x \in (0, q_0)$ $p'_+ (x)$ and $p'_- (x)$ are respectively right-continuous and left-continuous and $p'_+ (x) \leq p'_- (x)$.

Now we show that p', the derivative of p, is continuous over $(0, q_0)$. We prove that there exists no $x \in (0, q_0)$ at which $p'_+ (x) < p'_- (x)$. Let us admit that $p'_+ (\hat{x}) < p'_- (\hat{x})$ at $\hat{x} \in (0, q_0)$. We know that there exists a compact interval $I_{\hat{x}} \subset \mathbb{R}_{++}$ such that $\hat{x} \in \mathring{I}_{\hat{x}} \neq \emptyset$, p is Lipschitz continuous over $I_{\hat{x}}$, p'_+ and p'_- are defined over $I_{\hat{x}}$. Let τ be the Lipschitz constant for p over $I_{\hat{x}}$. Clearly τ is positive. Let $\nu \geq 2$ be an integer such that $4\hat{x}/\nu < \min \{\hat{x} - \min I_{\hat{x}}, \max I_{\hat{x}} - \hat{x}\}$ and $-4\tau\hat{x}/\nu + p (\hat{x}) > 0$. Let $\alpha > 0$ be a real such that

$$p'_- (\hat{x}) \, \hat{x}/\nu + p (\hat{x}) - \alpha > 0 > p'_+ (\hat{x}) \, \hat{x}/\nu + p (\hat{x}) - \alpha. \tag{1}$$

There must exist a positive value $\varepsilon < \hat{x}/\nu$ such that

$$p'_- (\hat{x}) \, (\hat{x}/\nu + \varepsilon) + p (\hat{x}) - \alpha > 0 > p'_+ (\hat{x}) \, (\hat{x}/\nu + \varepsilon) + p (\hat{x}) - \alpha, \tag{2}$$

and

$$p'_{-}(\hat{x})(\hat{x}/\nu-\varepsilon)+p(\hat{x})-\alpha>0>p'_{+}(\hat{x})(\hat{x}/\nu-\varepsilon)+p(\hat{x})-\alpha. \quad (3)$$

Since $-4\tau\hat{x}/\nu+p(\hat{x})>0$ then $-\tau(\hat{x}/\nu+r)+p(\hat{x})>0$, $\forall r\in[-3\hat{x}/\nu,3\hat{x}/\nu]$. This entails that:

$$-\tau r+p\left(\hat{x}\frac{\nu-1}{\nu}+r\right)>0\ \forall r\in[0,\hat{x}/\nu],$$

$$-\tau r+p\left(\hat{x}\frac{\nu-1}{\nu}+\varepsilon+r\right)>0\ \forall r\in[0,\hat{x}/\nu-\varepsilon],$$

$$-\tau r+p\left(\hat{x}\frac{\nu-1}{\nu}-\varepsilon+r\right)>0\ \forall r\in[0,\hat{x}/\nu+\varepsilon].$$

Since $p'_{+}(x)\geq-\tau$, $\forall x\in\left[\hat{x}-2\frac{\hat{x}}{\nu},\hat{x}+2\frac{\hat{x}}{\nu}\right]$:

$$p'_{+}\left(\hat{x}\frac{\nu-1}{\nu}+r\right)r+p\left(\hat{x}\frac{\nu-1}{\nu}+r\right)>0\forall r\in[0,\hat{x}/\nu], \quad (4)$$

$$p'_{+}\left(\hat{x}\frac{\nu-1}{\nu}+\varepsilon+r\right)r+p\left(\hat{x}\frac{\nu-1}{\nu}+\varepsilon+r\right)>0\forall r\in[0,\hat{x}/\nu-\varepsilon], \quad (5)$$

$$p'_{+}\left(\hat{x}\frac{\nu-1}{\nu}-\varepsilon+r\right)r+p\left(\hat{x}\frac{\nu-1}{\nu}-\varepsilon+r\right)>0\forall r\in[0,\hat{x}/\nu+\varepsilon]. \quad (6)$$

By (4) we have that $p\left(\hat{x}\frac{\nu-1}{\nu}+q\right)q$ is increasing over $[0,\hat{x}/\nu]$, hence $q_{\mu}\left(\hat{x}\frac{\nu-1}{\nu}\right)\geq\hat{x}/\nu$ and $p\left(\hat{x}\frac{\nu-1}{\nu}+q\right)q$ is strictly concave in q over $\left(0,q_{\mu}\left(\hat{x}\frac{\nu-1}{\nu}\right)\right)$. By[20] (1) we have that

$$\hat{x}/\nu\in\underset{q\in[0,\hat{x}/\nu]}{\arg\max}\,p\left(\hat{x}\frac{\nu-1}{\nu}+q\right)q-\alpha q. \quad (7)$$

By (5) we have that $p\left(\hat{x}\frac{\nu-1}{\nu}+\varepsilon+q\right)q$ is increasing over $[0,\hat{x}/\nu-\varepsilon]$, therefore $q_{\mu}\left(\hat{x}\frac{\nu-1}{\nu}+\varepsilon\right)\geq\hat{x}/\nu-\varepsilon$ and $p\left(\hat{x}\frac{\nu-1}{\nu}+\varepsilon+q\right)q$ is strictly concave in q over $\left(0,q_{\mu}\left(\hat{x}\frac{\nu-1}{\nu}+\varepsilon\right)\right)$. By (2) we have that

$$\hat{x}/\nu-\varepsilon\in\underset{q\in[0,\hat{x}/\nu+\varepsilon]}{\arg\max}\ p\left(\hat{x}\frac{\nu-1}{\nu}+\varepsilon+q\right)q-\alpha q. \quad (8)$$

[20] For the proof of the validity of (7), (8), and (9), consider the following statement:

Let $f:R_{+}\to R_{+}$ be a continuous function such that $\underset{x\in\mathbb{R}_{+}}{\arg\max}f(x)\neq\emptyset$ and $\eta:=\min\underset{x\in\mathbb{R}_{+}}{\arg\max}f(x)\neq0$. Let $\beta\in R_{++}$. If $f(x)$ is concave and increasing in x over $(0,\eta)$, then at every $\hat{x}\in(0,\eta)$ such that $f'_{+}(\hat{x})-\beta\leq0\leq f'_{-}(\hat{x})-\beta$ we have that $\hat{x}\in\underset{x\in\mathbb{R}_{+}}{\arg\max}f(x)-\beta x$, therefore $\hat{x}\in\underset{x\in[0,t]}{\arg\max}f(x)-\beta x$ for every $t\geq\hat{x}$.

Its proof is very simple: consider the properties of concave functions and that $\forall x>\eta$ we have $f(x)-\beta x<f(\eta)-\beta\eta$.

By (6) we have that $p\left(\hat{x}\frac{\nu-1}{\nu}-\varepsilon+q\right)q$ is increasing over $[0,\hat{x}/\nu+\varepsilon]$, therefore $q_\mu\left(\hat{x}\frac{\nu-1}{\nu}-\varepsilon\right)\geq\hat{x}/\nu+\varepsilon$ and $p\left(\hat{x}\frac{\nu-1}{\nu}-\varepsilon+q\right)q$ is strictly concave in q over $\left(0,q_\mu\left(\hat{x}\frac{\nu-1}{\nu}-\varepsilon\right)\right)$. By (3) we have that

$$\hat{x}/\nu+\varepsilon\in\underset{q\in[0,\hat{x}/\nu+\varepsilon]}{\arg\max}\ p\left(\hat{x}\frac{\nu-1}{\nu}-\varepsilon+q\right)q-\alpha q. \tag{9}$$

Now let us consider $o:=(p,\nu,(K_1,...,K_v),(c_1,..,c_\nu))$ where $K_i=[0,\hat{x}/\nu]$ for $i=1,...,\nu-2$ and $K_i=[0,\hat{x}/\nu+\varepsilon]$ for $i=\nu-1,\nu$ and $c_i:K_i\to\mathbb{R}_+$,$c_i:x\mapsto\alpha x$, $\forall i=1,..,\nu$. We show that $\mathbf{e}_1:=(\underbrace{\hat{x}/\nu,...,\hat{x}/\nu}_{\nu-2},\hat{x}/\nu-\varepsilon,\hat{x}/\nu+\varepsilon)$ and $\mathbf{e}_2:=(\underbrace{\hat{x}/\nu,...,\hat{x}/\nu}_{\nu-2},\hat{x}/\nu+\varepsilon,\hat{x}/\nu-\varepsilon)$ are two different Cournot equilibria for o.

Let us consider $\mathbf{e}_1$. For each firm $i=1,..,\nu-2$ the aggregate production of the other firms is $\hat{x}\frac{\nu-1}{\nu}$, by (7) we know that $\hat{x}/\nu$ maximizes firm i's profit function when the aggregate production of the other firms is $\hat{x}\frac{\nu-1}{\nu}$. For $i=\nu-1$ the aggregate production of the other firms is $\hat{x}\frac{\nu-1}{\nu}-\varepsilon$, by (8) we know that $\hat{x}/\nu+\varepsilon$ maximizes firm i's profit function when the aggregate production of the other firms is $\hat{x}\frac{\nu-1}{\nu}-\varepsilon$. For $i=\nu$ the aggregate production of the other firms is $\hat{x}\frac{\nu-1}{\nu}+\varepsilon$, by (9) we know that $\hat{x}/\nu-\varepsilon$ maximizes firm i's profit function when the aggregate production of the other firms is $\hat{x}\frac{\nu-1}{\nu}+\varepsilon$; therefore $\mathbf{e}_1$ is a Cournot equilibrium. By symmetric reasoning it can been shown that $\mathbf{e}_2$ is an equilibrium. Since $\varepsilon\neq 0$ then $\mathbf{e}_1\neq\mathbf{e}_2$.

Therefore the differentiability of p over $(0,q_0)$ is a necessary condition. Let p' be the derivative of p. Since $p'(x)=p'_+(x)=p'_-(x)$ for every $x\in(0,q_0)$, p'_+ is right-continuous over $(0,q_0)$ and p'_- is left-continuous over $(0,q_0)$, then p' must be continuous over $(0,q_0)$.

Hence whenever p is not continuously differentiable over $(0,q_0)$, we can construct an oligopoly in $O^p_{lin(2)}$, with two different equilibria. ■

Proposition 11 *Let $p:\mathbb{R}_+\to\mathbb{R}_+$ be a price function, continuous over $\mathbb{R}_{++}$, null at some point of $\mathbb{R}_{++}$, such that $p(0)=\limsup_{x\to0^+}p(x)$ and $\liminf_{x\to0^+}p(x)>0$. Every oligopoly in $O^p_{lin(2)}$ has a unique Cournot equilibrium only if $p(q+z)q$ is quasiconcave in q over $\mathbb{R}_+$, $\forall z\in\mathbb{R}_+$.*

Proof. Let $q_0:=\min\{x\in\mathbb{R}_{++}:p(x)=0\}$. Let $q_\mu:[0,q_0)\to\mathbb{R}$, $q_\mu:z\mapsto\min\arg\max_{x\in[0,q_0]}p(z+x)x$. By the previous propositions p must be nonincreasing over $\mathbb{R}_+$, continuously differentiable and decreasing over $(0,q_0)$, and $p(q+z)q$ must be strictly concave in q over $(0,q_\mu(z))$, $\forall z\in[0,q_0)$. If $z\geq q_0$, since $p(x+z)=0$ $\forall x\in\mathbb{R}_+$, $p(q+z)q$ is quasiconcave in q over $\mathbb{R}_+$. If $z\in[0,q_0)$, then $p(q+z)q$ is not quasiconcave in q over $\mathbb{R}_+$ only if there exists a value $\hat{q}\in(q_\mu(z),q_0-z)$ such that $p'(\hat{q}+z)\hat{q}+p(\hat{q}+z)>0$[21].

[21] If A is a compact real interval with nonempty interior and a continuous function $f:A\to\mathbb{R}$ is quasiconcave over A then f is nonincreasing over any interval $B\subseteq[\min\arg\max_{q\in A}f(q),\max A]$.

Let us admit that there exist two values $z \in [0, q_0)$ and $\hat{q} \in (q_\mu(z), q_0 - z)$ such that $p'(\hat{q}+z)\hat{q} + p(\hat{q}+z) > 0$. Let $\alpha \in (0, p'(\hat{q}+z)\hat{q} + p(\hat{q}+z))$. Let $m_1 : [0, \hat{q}] \to \mathbb{R}$, $m_2 : [0, \hat{q}] \to \mathbb{R}$, and $m_3 : [0, \hat{q}] \to \mathbb{R}$, such that $m_1 : k \mapsto \max_{q \in [0, \hat{q}-k]} p(q+k+z)q - \alpha q$, $m_2 : k \mapsto \max_{q \in [\hat{q}-k, q_0-k]} p(q+k+z)q - \alpha q$, and $m_3 : k \mapsto \max_{q \in [0, q_0-k]} p(q+k+z)q - \alpha q$. Since

$$\max_{q \in [0,\hat{q}]} p(q+z)q - \alpha q = p(q_\mu(z)+z)q_\mu(z) - \alpha q_\mu(z),$$

and since[22]

$$p(q_\mu(z)+z)q_\mu(z) - \alpha q_\mu(z) > p(q+z)q - \alpha q \quad \forall q \in [\hat{q}, q_0]$$

then

$$\max_{q \in [0,\hat{q}]} p(q+z)q - \alpha q > \max_{q \in [\hat{q}, q_0]} p(q+z)q - \alpha q.$$

Hence $m_1(0) > m_2(0)$, since

$$m_1(\hat{q}) = p(0+\hat{q}+z)0 - \alpha 0 = 0$$

and[23]

$$m_2(\hat{q}) > 0.$$

Hence $m_2(\hat{q}) > m_1(\hat{q})$. By Berge's maximum theorem m_1,m_2 and m_3 are continuous, since $m_1(0) > m_2(0)$ and $m_1(\hat{q}) < m_2(\hat{q})$, then there exists a value, say $\bar{k} \in (0, \hat{q})$, such that $m_1(\bar{k}) = m_2(\bar{k}) = m_3(\bar{k})$. Since $p'(\hat{q}+z) \leq 0$, then

$$p'(\hat{q}+z)(\hat{q}-\bar{k}) + p(\hat{q}+z) - \alpha \geq p'(\hat{q}+z)\hat{q} + p(\hat{q}+z) - \alpha > 0,$$

this entails that $\hat{q} - \bar{k}$ does not maximize $p(q+\bar{k}+z)q - \alpha q$ over $[0, q_0 - \bar{k}]$. Let $y := \bar{k} + z > 0$. Let $b := q_0 - \bar{k} > 0$. We can conclude that there exist two values x_1 and x_2 such that $x_2 > x_1$ and

$$p(x_1+y)x_1 - \alpha x_1 \geq p(q+y)q - \alpha q \quad \forall q \in [0, b], \tag{1}$$

$$p(x_2+y)x_2 - \alpha x_2 \geq p(q+y)q - \alpha q \quad \forall q \in [0, b]. \tag{2}$$

Since $y > 0$ and $x_2 + y < q_0$ let $I \supset [0, x_2+y]$ be a compact interval such that $y > \min I > 0$ and $q_0 > \max I > x_2 + y$. Let τ be the Lipschitz constant of p over I. Let $\beta \in (0, p(x_2+y))$. Let $\nu \geq 1$ be an integer such that $\frac{y}{\nu} < y - \min I$ and $-\tau x + p(x_2+y) - \beta > 0$ $\forall x \in \left[0, \frac{y}{\nu}\right]$. Since $-\tau q \leq p'\left(x_1 + y\frac{\nu-1}{\nu} + q\right)q$ $\forall q \in \left[0, \frac{y}{\nu}\right]$, $-\tau q \leq p'\left(x_2 + y\frac{\nu-1}{\nu} + q\right)q$ $\forall q \in \left[0, \frac{y}{\nu}\right]$ and $p(x_2+y) \leq p\left(x_2 + y\frac{\nu-1}{\nu} + q\right) \leq p\left(x_1 + y\frac{\nu-1}{\nu} + q\right)$ $\forall q \in \left[0, \frac{y}{\nu}\right]$, then

$$p'\left(x_1 + y\frac{\nu-1}{\nu} + q\right)q + p\left(x_1 + y\frac{\nu-1}{\nu} + q\right) - \beta > 0 \ \forall q \in \left[0, \frac{y}{\nu}\right],$$

[22] Recall that $\hat{q} > q_\mu(z)$.

[23] Notice that $p(0+\hat{q}+z)0 - \alpha 0 = 0$ and $p'(0+\hat{q}+z)0 + p(0+\hat{q}+z) - \alpha > 0$.

$$p'\left(x_2+y\frac{\nu-1}{\nu}+q\right)q+p\left(x_2+y\frac{\nu-1}{\nu}+q\right)-\beta>0\ \forall q\in\left[0,\frac{y}{\nu}\right].$$

Therefore

$$p\left(x_1+y\right)\frac{y}{\nu}-\beta\frac{y}{\nu}\geq p\left(x_1+y\frac{\nu-1}{\nu}+q\right)q-\beta q\quad\forall q\in\left[0,\frac{y}{\nu}\right],\tag{3}$$

$$p\left(x_2+y\right)\frac{y}{\nu}-\beta\frac{y}{\nu}\geq p\left(x_2+y\frac{\nu-1}{\nu}+q\right)q-\beta q\quad\forall q\in\left[0,\frac{y}{\nu}\right].\tag{4}$$

Now let us consider the oligopoly $o:=\left(p,\nu+1,\left(K_1,...,K_{v+1}\right),\left(c_1,..,c_{\nu+1}\right)\right)$ where $K_i=\left[0,\frac{y}{\nu}\right]$ for $i=1,...,\nu$ and $K_{\nu+1}=[0,b]$, and $c_i:K_i\to\mathbb{R}_+$, $c_i:x\mapsto\beta x$, $\forall i=1,..,\nu$ and $c_{\nu+1}:K_{\nu+1}\to\mathbb{R}_+$, $c_{\nu+1}:x\mapsto\alpha x$. We show that $\mathbf{e}_1:=\left(y/\nu,...,y/\nu,x_1\right)$ and $\mathbf{e}_2:=\left(y/\nu,...,y/\nu,x_2\right)$ are different Cournot equilibria for $o\in O^p_{lin(2)}$.

Let us consider $\mathbf{e}_1$. For each firm $i=1,..,\nu$ the aggregate production of the other firms is $y\frac{\nu-1}{\nu}+x_1$, by (3) we know that y/ν maximizes the profit function of firm i when the aggregate production of the other firms is $y\frac{\nu-1}{\nu}+x_1$. For firm $\nu+1$ the aggregate production of the other firms is y, by (1) we know that x_1 maximizes the profit function of firm $\nu+1$ when the aggregate production of the other firms is y. Therefore $\mathbf{e}_1$ is a Cournot equilibrium. Considering (4) and (2) we can show that $\mathbf{e}_2$ is an equilibrium.

Hence whenever $p\left(q+z\right)q$ is not quasiconcave in q over $\mathbb{R}_+$, $\forall z\in\mathbb{R}_+$ we can construct an oligopoly in $O^p_{lin(2)}$ with two different equilibria. ■

Proposition 12 *Let $p:\mathbb{R}_+\to\mathbb{R}_+$ be a price function, continuous over $\mathbb{R}_{++}$, null at some point of $\mathbb{R}_{++}$, such that $p\left(0\right)=\limsup_{x\to0^+}p\left(x\right)$ and $\liminf_{x\to0^+}p\left(x\right)>0$. Let $q_0:=\min\left\{x\in\mathbb{R}_{++}:p\left(x\right)=0\right\}$, and let us assume that p is continuously differentiable over $\left(0,q_0\right)$. Every oligopoly in $O^p_{lin(2)}$ has a unique Cournot equilibrium only if for every integer $m\geq1$, $\forall x\in\left(0,q_0\right)$,$\forall y\in[0,x]$,$\forall z\in\left(0,q_0-x\right)$, $p'\left(x\right)y+mp\left(x\right)>0$ entails that $p'\left(x\right)y+mp\left(x\right)>p'\left(x+z\right)\left(y+z\right)+mp\left(x+z\right)$.*

Proof. Let $q_\mu:[0,q_0)\to\mathbb{R}$, $q_\mu:z\mapsto\min\arg\max_{x\in[0,q_0]}p\left(z+x\right)x$. By the previous propositions p must be nonincreasing over $\mathbb{R}_+$, decreasing over $\left(0,q_0\right)$, and $p\left(q+z\right)q$ must be strictly concave in q over $\left(0,q_\mu\left(z\right)\right)$, $\forall z\in[0,q_0)$ and quasiconcave in q over $\mathbb{R}_+$, $\forall z\in\mathbb{R}_+$.

Let us suppose by absurd that there exist four values $m\geq1$, $x\in\left(0,q_0\right)$, $y\in[0,x]$, $h\in\left(0,q_0-x\right)$ such that $p'\left(x+h\right)\left(y+h\right)+mp\left(x+h\right)\geq p'\left(x\right)y+mp\left(x\right)>0$. It is impossible that $y=0$ since we would have $p'\left(x+h\right)h+mp\left(x+h\right)\geq mp\left(x\right)$. It is impossible that $m=1$ since we would have $0<p'\left(x\right)y+p\left(x\right)\leq p'\left(x+h\right)\left(y+h\right)+p\left(x+h\right)$: since $p\left(q+z\right)q$ must be strictly concave in q over $\left(0,q_\mu\left(z\right)\right)$, $\forall z\in[0,q_0)$ and quasiconcave in q over $\mathbb{R}_+$, $\forall z\in\mathbb{R}_+$ then $p'\left(x\right)y+p\left(x\right)>0$ entails that $p'\left(x+h\right)h+p\left(x+h\right)<p'\left(x\right)y+p\left(x\right)$. Therefore let us assume that $y>0$ and $m\geq2$. By absurd: we would have that

$$p'\left(x+h\right)\frac{\left(y+h\right)}{m}+p\left(x+h\right)\geq p'\left(x\right)\frac{y}{m}+p\left(x\right)>0.$$

Let $\alpha := \frac{p'(x)\frac{y}{m}+p(x)}{2} > 0$. We must have that

$$p'(x)\frac{y}{m} + p(x) - \alpha > 0 \tag{1}$$

and

$$p'(x+h)\frac{(y+h)}{m} + p(x+h) - \alpha > 0. \tag{2}$$

Since $p(q+z)q$ must be quasiconcave in q over $\mathbb{R}_+$, $\forall z \in \mathbb{R}_+$, $p(q+z)q$ must be strictly concave in q over $(0, q_\mu(z))$, $\forall z \in [0, q_0)$ and $\alpha > 0$, then $p(q+z)q - \alpha q$ must be strictly quasiconcave in q over $\mathbb{R}_+$, $\forall z \in \mathbb{R}_+$; in particular $p\left(q + \left(x - \frac{y}{m}\right)\right)q - \alpha q$ must be strictly quasiconcave in q over $\left[0, \frac{y+h}{m}\right]$ and $p\left(q + \left(x + h - \frac{y+h}{m}\right)\right)q - \alpha q$ must be strictly quasiconcave in q over $\left[0, \frac{y+h}{m}\right]$.

Therefore (1) is sufficient to affirm that, $\forall q \in \left[0, \frac{y+h}{m}\right]$,

$$p(x)\frac{y}{m} - \alpha\frac{y}{m} \geq p\left(q + \left(x - \frac{y}{m}\right)\right)q - \alpha q; \tag{3}$$

and (2) is sufficient to affirm that, $\forall q \in \left[0, \frac{y+h}{m}\right]$,

$$p(x+h)\frac{y+h}{m} - \alpha\frac{y+h}{m} \geq p\left(q + \left(x + h - \frac{y+h}{m}\right)\right)q - \alpha q. \tag{4}$$

If $y = x$ let us consider $\bar{o} := (p, m, (K_1, ..., K_m), (c_1, .., c_m))$ where $K_i = \left[0, \frac{y+h}{m}\right]$ for $i = 1, .., m$, and $c_i : K_i \to \mathbb{R}_+$, $c_i : x \mapsto \alpha x$, $\forall i = 1, .., m$. Lets us show that $\mathbf{e}_1 := \left(\frac{y}{m}, .., \frac{y}{m}\right)$ and $\mathbf{e}_2 := \left(\frac{y+h}{m}, .., \frac{y+h}{m}\right)$ are two different Cournot equilibria for $\bar{o} \in O^p_{lin(2)}$. Since $h > 0$ then $\mathbf{e}_2 \neq \mathbf{e}_1$. Let us consider $\mathbf{e}_1$: for each firm $i = 1, .., m$ the aggregate production of the other firms is $(m-1)\frac{y}{m} = x - \frac{y}{m}$, by (3) we know that $\frac{y}{m}$ maximizes the profit function of firm i when the aggregate production of the other firms is $(m-1)\frac{y}{m}$; therefore $\mathbf{e}_1$ is a Cournot equilibrium. Considering (4) we can similarly show that $\mathbf{e}_2$ is an equilibrium.

If $y < x$ let $\nu \geq 1$ be an integer such that

$$p'(x)\frac{x-y}{\nu} + p(x) - \alpha > 0 \tag{5}$$

and

$$p'(x+h)\frac{x-y}{\nu} + p(x+h) - \alpha > 0. \tag{6}$$

Since $p(q+z)q$ must be quasiconcave in q over $\mathbb{R}_+$, $\forall z \in \mathbb{R}_+$, $p(q+z)q$ must be strictly concave in q over $(0, q_\mu(z))$, $\forall z \in [0, q_0)$ and $\alpha > 0$ then $p(q+z)q - \alpha q$ must be strictly quasiconcave in q over $\mathbb{R}_+$, $\forall z \in \mathbb{R}_+$; in particular $p\left(q + y + (\nu - 1)\frac{x-y}{\nu}\right)q - \alpha q$ must be strictly quasiconcave in q over $\left[0, \frac{x-y}{\nu}\right]$ and $p\left(q + y + h + (\nu - 1)\frac{x-y}{\nu}\right)q - \alpha q$ must be strictly quasiconcave in q over $\left[0, \frac{x-y}{\nu}\right]$.

Therefore (5) is sufficient to affirm that, $\forall q \in \left[0, \frac{x-y}{\nu}\right]$,

$$p(x)\frac{x-y}{\nu} - \alpha\frac{x-y}{\nu} \geq p\left(q + y + (\nu - 1)\frac{x-y}{\nu}\right)q - \alpha q; \tag{7}$$

and (6) is sufficient to affirm that, $\forall q \in \left[0, \frac{x-y}{\nu}\right]$,

$$p(x+h)\frac{x-y}{\nu} - \alpha\frac{x-y}{\nu} \geq p\left(q + y + h + (\nu - 1)\frac{x-y}{\nu}\right)q - \alpha q. \tag{8}$$

Now let us consider the oligopoly $o := (p, \nu + m, (K_1, ..., K_{v+m}), (c_1, .., c_{\nu+m}))$ where $K_i = \left[0, \frac{x-y}{\nu}\right]$ for $i = 1, ..., \nu$ and $K_i = \left[0, \frac{y+h}{m}\right]$ for $i = \nu + 1, .., \nu + m$, and $c_i : K_i \to \mathbb{R}_+$, $c_i : x \mapsto \alpha x$, $\forall i = 1, .., \nu + m$. We show that $\mathbf{e}_1 := (\underbrace{\frac{x-y}{\nu}, .., \frac{x-y}{\nu}}_{\nu}, \underbrace{\frac{y}{m}, .., \frac{y}{m}}_{m})$ and $\mathbf{e}_2 := (\underbrace{\frac{x-y}{\nu}, .., \frac{x-y}{\nu}}_{\nu}, \underbrace{\frac{y+h}{m}, .., \frac{y+h}{m}}_{m})$ are two different Cournot equilibria for $o \in O^p_{lin(2)}$.

Since $h > 0$, then $\mathbf{e}_2 \neq \mathbf{e}_1$. Let us consider $\mathbf{e}_1$. For each firm $i = 1, .., \nu$ the aggregate production of the other firms is $y + (\nu - 1)\frac{x-y}{\nu}$, by (7) we know that $\frac{x-y}{\nu}$ maximizes the profit function of firm i when the aggregate production of the other firms is $y + (\nu - 1)\frac{x-y}{\nu}$. For each firm $i = \nu + 1, .., \nu + m$ the aggregate production of the other firms is $x - \frac{y}{m}$, by (3) we know that $\frac{y}{m}$ maximizes the profit function of firm i when the aggregate production of the other firms is $x - \frac{y}{m}$. Hence $\mathbf{e}_1$ is a Cournot equilibrium. By (8) and (4) we can similarly show that $\mathbf{e}_2$ is an equilibrium.

Therefore if p is differentiable over $(0, q_0)$ and for some integer $m \geq 1$, for some $x \in (0, q_0)$, for some $y \in [0, x]$, for some $h \in (0, q_0 - x)$, we have $p'(x)y + mp(x) > 0$ and $p'(x)y + mp(x) > p'(x+h)(y+h) + mp(q+h)$ then we can construct an oligopoly in $O^p_{lin(2)}$ with two different equilibria. ■

Proposition 13 *Let $p : \mathbb{R}_+ \to \mathbb{R}_+$ be a price function, continuous over $\mathbb{R}_{++}$, null at some point of $\mathbb{R}_{++}$, such that $p(0) = \limsup_{x\to 0^+} p(x)$ and $\liminf_{x\to 0^+} p(x) > 0$. Let $q_0 := \min\{x \in \mathbb{R}_{++} : p(x) = 0\}$ and $q_\mu : [0, q_0) \to \mathbb{R}$, $q_\mu : z \mapsto \min \arg\max_{x\in[0,q_0]} p(z+x)x$. Every oligopoly in $O^p_{lin(2)}$ has a unique Cournot equilibrium only if:*

1. p is continuously differentiable over $(0, q_0)$, decreasing over $(0, q_0)$ and nonincreasing over $\mathbb{R}_+$,
2. $p(q+z)q$ is strictly concave in q over $(0, q_\mu(z))$, $\forall z \in [0, q_0)$,
3. $p(q+z)q$ is quasiconcave in q over $\mathbb{R}_+$, $\forall z \in \mathbb{R}_+$,
4. for every integer $m \geq 1$, $\forall x \in (0, q_0)$,$\forall y \in [0, x]$,$\forall z \in (0, q_0 - x)$, $p'(x)y + mp(x) > 0$ entails that $p'(x)y + mp(x) > p'(x+z)(y+z) + mp(x+z)$.

Proof. An immediate consequence of the previous propositions. ■

Appendix B

Proposition 14 *Let $p : \mathbb{R}_+ \to \mathbb{R}_+$ be a price function, continuous over $\mathbb{R}_{++}$, null at some point of $\mathbb{R}_{++}$, such that $p(0) = \limsup_{x \to 0^+} p(x)$ and $\liminf_{x \to 0^+} p(x) > 0$. Let $q_0 := \min \{x \in \mathbb{R}_{++} : p(x) = 0\}$ and $q_\mu : [0, q_0) \to \mathbb{R}$, $q_\mu : z \mapsto \min \arg\max_{x \in [0,q_0]} p(z + x)\, x$. p is decreasing over $[0, q_0]$, nonincreasing over $\mathbb{R}_+$ and in particular $p'(x) < 0 \ \forall x \in (0, q_0)$ and $p(x) = 0 \ \forall x \geq q_0$, if:*

1. *p is continuously differentiable over $(0, q_0)$,*
2. *$p(q + z)\, q$ is strictly concave in q over $(0, q_\mu(z))$, $\forall z \in [0, q_0)$,*
3. *$p(q + z)\, q$ is quasiconcave in q over $\mathbb{R}_+$, $\forall z \in \mathbb{R}_+$.*

Proof. Since $p(q)\, q$ is a quasiconcave function then it must be nonincreasing over $[q_\mu(0), +\infty)$, therefore $p(x) = 0 \ \forall x \geq q_0$ and $p'(x)\, x + p(x) \leq 0 \ \forall x \in [q_\mu(0), q_0)$, hence $p'(x) \leq -\frac{p(x)}{x} < 0 \ \forall x \in [q_\mu(0), q_0)$. Since $p(q)\, q$ is strictly concave in q over $(0, q_\mu(0))$, then $p(q)\, q$ is strictly concave in q over $[0, q_\mu(0))$, this entails that $p'(x)\, x + p(x) < \frac{p(x)x - p(0)0}{x - 0} = p(x) \ \forall x \in (0, q_\mu(0))$, hence $p'(x) < 0 \ \forall x \in (0, q_\mu(0))$. This entails that p is decreasing over $[0, q_0]$ and nonincreasing over $\mathbb{R}_+$, and that $p'(x) < 0 \ \forall x \in (0, q_0)$. ■

4 - Sufficient Conditions

In this section we shall present necessary and sufficient conditions for the existence of a unique Cournot equilibrium. More specifically we shall prove that the conditions of Proposition 5 suffice for the existence of a unique Cournot equilibrium for every oligopoly in O^p_{cvx}. As we have already noted in the previous section the oligopolistic context described in the first pages of this work is particularly suitable for modelling problems of merger. Up to now we have not provided any definition of merger, therefore we must first clarify what we mean by that.

Let F be a set of n firms producing the same commodity and $c_i : D_i \, (\neq \emptyset) \subseteq \mathbb{R}_+ \to \mathbb{R}$ be the cost function of a generic firm $i \in F$. Let us assume that, $\forall i \in F$, D_i is closed and c_i is lower semicontinuous. Let $I \subseteq F$ be any set with $m > 0$ firms. From now on we shall assume that the merger of all the firms in I generates a new firm with productive capacity $D := \{ \sum_{i \in I} x_i : x_i \in D_i \}$ and cost function

$$c : D \to \mathbb{R}, \ c : z \mapsto \min_{(x_1, .., x_m) \in \prod_{i \in I} D_i : x_1 + .. + x_m = z} c_1(x_1) + .. + c_m(x_m).$$

This assumption is quite neutral since, according to it, a merger does not alter positively or negatively the cost functions of the firms involved in the merger operation, and the post-merger firm can just allocate its production across the plants of the merged firms[24]. Consistently with the rest of our work we shall assume that in case of merger the price function does not change.

It is plain that in a problem of entry, or in a problem of technological choice, the set of the possible oligopolies is a subset of O^p_{cvx} if each firm's set of technological choices consists of a class of production functions associated with convex, continuous and increasing cost functions which are null at zero. It is not immediate that all the possible mergers within each oligopoly in O^p_{cvx} result in oligopolies still belonging to O^p_{cvx}. We shall prove that this is actually the case; the reader may notice that, as a logical consequence, every proposition which is true for every oligopoly in O^p_{cvx} is necessarily true for the oligopolies generated by all the possible mergers within any oligopoly in O^p_{cvx}.

Before providing the formal proof we show that this *closure property*[25] may not hold for other classes of oligopolies. For instance a merger between the firms in a duopoly in O^p_{lin} may create a monopoly which does not belong to O^p_{lin}. Let us consider two firms, say A and B. The cost function of A is $c_A : [0,1] \to \mathbb{R}$, $c_A : x \mapsto x$, the cost function of B is $c_B : [0,1] \to \mathbb{R}$, $c_B : x \mapsto 2x$. A and B merge, thus forming a new firm, say AB. As above, let us assume that the cost function of AB is $c_{AB} : [0,2] \to \mathbb{R}$, $c_{AB} : z \mapsto \min_{(x_A,x_B)\in[0,1]\times[0,1]:x_A+x_B=z} c_A(x_A)+c_B(x_B)$. Hence $c_{AB} : x \mapsto \max\{x, 2x-1\}$. Notice that c_{AB} is convex, but not linear, over its domain, and that, while c_A and c_B have differentiable extensions, c_{AB} has not because it is not differentiable at $1 \in (0,2)$. Therefore a proposition which is true for every oligopoly in O^p_{lin} may be false for an oligopoly generated by some merger within an oligopoly in O^p_{lin}.

The following proposition is necessary to prove that every possible merger within each oligopoly in O^p_{cvx} generates an oligopoly in O^p_{cvx}.

Definition 15 *Let Ω be the set of real-valued functions whose domain is a closed real interval with minimum 0 and nonempty interior. Let $\Psi \subset \Omega$ be the set of functions in Ω which are continuous, increasing, convex and null at 0. Let Λ be the set of all nonempty subsets of Ψ with finite cardinality. Let $m : \Lambda \to \Omega$ be a function such that for every $\lambda = \{c_1,..,c_n\} \in \Lambda$:*

$$m(\lambda) : D \to \mathbb{R}_+, \quad m(\lambda) : z \mapsto \min_{(x_1,..,x_n)\in\prod_{i=1}^n D_i : x_1+..+x_n=z} c_1(x_1)+..+c_n(x_n),$$

where D_1,..,D_n are the domains of c_1,..,c_n and

$$D := \{y_1 + .. + y_n : y_1 \in D_1, .., y_n \in D_n\}.$$

Proposition 16 *Λ is closed under m.*

[24] Farrell and Shapiro (1990) name this type of merger "merger with no synergies".

[25] The reason for this name will soon become apparent.

Proof. We want to show that $m(\lambda) \in \Lambda$, $\forall \lambda \in \Lambda$. Let $\lambda := \{c_1, .., c_n\}$ be any element of Λ, and let D_1,..,D_n be the domains of c_1,..,c_n and $D := \{y_1 + .. + y_n : y_1 \in D_1, .., y_n \in D_n\}$. For notational convenience we denote $m(\lambda)$ by m_λ. It is immediate that D is a closed real interval with minimum 0 and non-empty interior. It is immediate that $m_\lambda(0) = 0$ and that $m_\lambda(x) \nleq 0 \; \forall x > 0$[26]. Continuity of m_λ over D simply derives from Berge's maximum theorem[27]. Let $(\check{z}, \hat{z})$ be any pair in $D \times D$ and let γ be any real in $(0, 1)$, to conclude we only need to prove that m_λ is a convex function, namely:

$$m_\lambda(\gamma\check{z} + (1-\gamma)\hat{z}) \leq \gamma m_\lambda(\check{z}) + (1-\gamma) m_\lambda(\hat{z}). \tag{1}$$

Let $\mathbf{a} = (a_1, .., a_n) \in \prod_{i=1}^n D_i$ be such that $\sum_{i=1}^n a_i = \check{z}$ and $\sum_{i=1}^n c_i(a_i) = m_\lambda(\check{z})$; let $\mathbf{b} = (b_1, .., b_n) \in \prod_{i=1}^n D_i$ be such that $\sum_{i=1}^n b_i = \hat{z}$ and $\sum_{i=1}^n c_i(b_i) = m_\lambda(\hat{z})$. It is immediate that $\gamma\mathbf{a}+(1-\gamma)\mathbf{b} \in \prod_{i=1}^n D_i$ and $\sum_{i=1}^n \gamma a_i+(1-\gamma) b_i = \gamma\check{z} + (1-\gamma)\hat{z}$. By convexity of c_1,..,c_n we have that $c_i(\gamma a_i + (1-\gamma) b_i) \leq \gamma c_i(a_i) + (1-\gamma) c_i(b_i)$, $\forall i = 1, .., n$, therefore

$$\sum_{i=1}^n c_i(\gamma a_i + (1-\gamma) b_i) \leq \sum_{i=1}^n \gamma c_i(a_i) + \sum_{i=1}^n (1-\gamma) c_i(b_i). \tag{2}$$

Since

$$m_\lambda(\gamma\check{z} + (1-\gamma)\hat{z}) \leq \sum_{i=1}^n c_i(\gamma a_i + (1-\gamma) b_i),$$

and

$$\sum_{i=1}^n \gamma c_i(a_i) + \sum_{i=1}^n (1-\gamma) c_i(b_i) = \gamma m_\lambda(\check{z}) + (1-\gamma) m_\lambda(\hat{z}),$$

by (2) we have that (1) is true. Therefore m_λ is a convex function. We can conclude that, since the range of m is contained in Λ, Λ is closed under m. ■

Let us now consider the set O^p_{cvx}. Let o be an oligopoly in O^p_{cvx}. If o consists of just one firm then the only possible merger generates anew the firm itself, which, by assumption, is an oligopoly in O^p_{cvx}. If o consists of many firms we have many possible mergers. Let $N = \{1, .., n\}$ be the set of firms of o and $P := \{P_1, .., P_l\}$ be a partition of N[28]. Let us assume that the firms of each

[26] Notice that these two last conditions entails that if m_λ is convex then it is also increasing over D.

[27] Berge's maximum theorem states:

Let $G \subseteq R^m$, $Y \subseteq R^k$ and let $\gamma : G \to Y$ be a compact-valued multi-valued function. Let $f : Y \to R$ be continuous function. Let $\mu : G \to Y$, $\mu : x \mapsto \arg\max_{\mathbf{y}\in\gamma(\mathbf{x})} f(\mathbf{y})$ be a multi-valued function. Let $F : G \to R$, $F : x \mapsto \max_{\mathbf{y}\in\gamma(\mathbf{x})} f(\mathbf{y})$ be a function. If γ is continuous at x then $\mu(\mathbf{x})$ is compact, μ is upper semicontinuous at x and F is continuous at x.

As to our proposition: let $G := D$ and $\Delta_v^{n-1} := \{\mathbf{y} \in \mathbb{R}_+^n : y_1 + .. + y_n = v\}$,$\forall v \in \mathbb{R}_+$, let $Y = \prod_{i=1}^n D_i$, $\gamma : z \mapsto \Delta_z^{n-1} \cap Y$, $f : \mathbf{x} \mapsto -c_1(x_1) - ... - c_n(x_n)$, simply apply the theorem and notice that the continuity of F entails the continuity of m_λ at every point of D.

[28] For instance, if $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$ we may consider the following partition $\{\{1, 3, 5\}, \{2\}, \{8\}, \{4, 6, 7\}\}$.

set P_i merge into a new firm ϕ_i, $\forall i = 1,..,l$. From the previous proposition it follows that the cost function of each new firm ϕ_i is a convex continuous increasing function, null at 0, defined over a closed interval with minimum 0 and nonempty interior. Hence the post-merger oligopoly consisting of firms ϕ_1, ϕ_2,...,ϕ_m belongs to O^p_{cvx}.

To understand the importance of this conclusion consider the case of a market with no incumbent firms and an indefinite number of potential entrants which can enter the market by choosing a technology from among a set of possible technologies associated with post-entry cost functions in Ψ. In this case, whatever the technology adopted, every oligopoly generated by the entry decisions of the potential entrants is an element of O^p_{cvx}, but most importantly, every variation in the number of active firms, whether it be determined by the exit of some firms, by a later entry of some other firms or by mergers, generates an oligopoly still belonging to O^p_{cvx}. Therefore, under the previous assumption on the technology of potential entrants, a proposition establishing sufficient conditions on p for the existence of a unique Cournot equilibrium for every oligopoly in O^p_{cvx} holds for all the possible oligopolies that can be generated by choices of entry, exit and merger.

As shown above, when dealing with upper bounded productive capacities we may obtain post-merger cost functions which are not differentiable at some point of the interior of the domain. For this reason we are not going to restrict to differentiable cost functions our analysis on sufficient conditions, even though we must assume continuous differentiability of the price function over the interior of its support: it should be clear from the previous section that this condition is necessary.

The reader might be tempted to think that if we had not considered upper bounded productive capacities we might have shown that sets of oligopolies with convex cost functions defined over $\mathbb{R}_+$ and a certain order of differentiability are closed under the operation of merger defined above. Once again we find something that, perhaps, is unexpected. Consider this case of two firms, say A and B, respectively with cost functions $c_A : \mathbb{R}_+ \to \mathbb{R}$, $c_A : x \mapsto x$ and $c_B : \mathbb{R}_+ \to \mathbb{R}$, $c_B : x \mapsto x^2$. These cost functions can be easily extended to $\mathbb{R}$ as infinitely differentiable functions, yet, when A and B merge, the post-merger cost function $c : \mathbb{R}_+ \to \mathbb{R}_+$, $c : x \mapsto \begin{cases} x^2 & \text{if } x \in [0, 1/2] \\ x - 1/4 & \text{if } x \geq 1/2 \end{cases}$ can be extended to $\mathbb{R}$ as a continuously differentiable function but cannot be extended as a twice differentiable function[29]. It is important to remark this point, since, in general, a theorem on the existence of a unique Cournot equilibrium holding for a certain set of oligopolies need not hold for an oligopoly generated by a merger within an oligopoly of that set; this does not happen with O^p_{cvx}.

We recall that the cost functions of the oligopolies in O^p_{cvx} are defined over closed real intervals with minimum 0 and nonempty interiors, and are continuous, convex, increasing and null at 0. This entails that for each possible cost function the right-hand and the left-hand derivative are defined over the interior

[29] Since it is not twice differentiable at 1/2.

of the domain, but not in general over the whole domain. In particular when the domain is upper bounded a cost function may not possess the left-hand derivative at the maximum of its domain. Therefore we can also conclude that in O^p_{cvx} local Lipschitz continuity may fail at the maximum of the domains of the cost functions; though it certainly holds everywhere else.

We shall now extend our first result of the previous section to the whole set O^p_{cvx}.

Proposition 17 *Let $p : \mathbb{R}_+ \to \mathbb{R}_+$ be a price function, continuous over $\mathbb{R}_{++}$, null at some point of $\mathbb{R}_{++}$, such that $p(0) = \limsup_{x \to 0^+} p(x)$ and $\liminf_{x \to 0^+} p(x) > 0$. Let $q_0 := \min\{x \in R_{++} : p(x) = 0\}$ and $q_\mu : [0, q_0) \to \mathbb{R}$ be a function such that $q_\mu : z \mapsto \min \arg \max_{x \in [0,q_0]} p(z + x)\, x$. Let $(p, n, (K_1, .., K_n), (c_1, .., c_n)) \in O^p_{cvx}$. There exists a unique Cournot equilibrium for $(p, n, (K_1, .., K_n), (c_1, .., c_n))$ if:*

1. *$p(q + z)\, q$ is strictly concave in q over $(0, q_\mu(z))$, $\forall z \in [0, q_0)$,*
2. *$p(q + z)\, q$ and quasiconcave in q over $\mathbb{R}_+$,$\forall z \in \mathbb{R}_+$,*
3. *p is continuously differentiable over $(0, q_0)$,*
4. *for every integer $m \geq 1$, $\forall x \in (0, q_0)$,$\forall y \in [0, x]$,$\forall z \in (0, q_0 - x)$, $p'(x)\, y + mp(x) > 0$ entails that $p'(x)\, y + mp(x) > p'(x + z)(y + z) + mp(q + z)$.*

The formal proof is in *Appendix C*[30] and enables the computation of the unique Cournot equilibrium by simple approximation methods. The proof employs only elementary concepts of real analysis and does not involve any fixed point theorem for functions of more than one variable.

The reader may easily notice that, since $O^p_{lin(2)} \subset O^p_{cvx}$, if p is continuous over $\mathbb{R}_{++}$, null at some point of $\mathbb{R}_{++}$, $p(0) = \limsup_{x \to 0^+} p(x)$ and $\liminf_{x \to 0^+} p(x) > 0$ then, by Proposition 5, conditions 1 through 4 are also necessary. Therefore conditions 1 through 4 are necessary and sufficient.

It may be hard to verify whether a function satisfies conditions 1 through 4. The following proposition shows that if $p(q)\, q$ is differentiable and strictly concave over $(0, q_0)$ then conditions 1 through 4 are satisfied. Strict concavity is not a necessary condition, yet one can easily verify whether this condition is satisfied or not.

Proposition 18 *Let $p : \mathbb{R}_+ \to \mathbb{R}_+$ be a price function, continuous over $\mathbb{R}_{++}$, null at some point of $\mathbb{R}_{++}$, such that $p(0) = \limsup_{x \to 0^+} p(x)$ and $\liminf_{x \to 0^+} p(x) > 0$. Let $q_0 := \min\{x \in R_{++} : p(x) = 0\}$ and $q_\mu : [0, q_0) \to \mathbb{R}$ be a function such that $q_\mu : z \mapsto \min \arg \max_{x \in [0,q_0]} p(z + x)\, x$. If p is continuously differentiable over $(0, q_0)$, $p(q)\, q$ is strictly concave in q over $(0, q_0)$ and $p(q) = 0$ $\forall q \geq q_0$ then conditions 1 through 4 are satisfied.*

[30] The reader may easily notice that, since p is null for every value greater than q_0, the previous proposition cannot be extended to all the oligopolies with convex, continuous, non-decreasing cost functions null at 0.

The proof is in *Appendix D*. The reader should recall that, when p is continuous over $\mathbb{R}_{++}$, null at some point of $\mathbb{R}_{++}$, and such that $p(0) = \limsup_{x \to 0^+} p(x)$ and $\liminf_{x \to 0^+} p(x) > 0$, conditions 1 to 4 entail that the derivative of p is negative over $(0, q_0)$ and p is continuous over $\mathbb{R}_+$: this implication clearly holds for any set of functions which satisfies conditions 1 through 4.

Appendix C

Proposition 19 *Let $p : \mathbb{R}_+ \to \mathbb{R}_+$ be a price function, continuous over $\mathbb{R}_{++}$, null at some point of $\mathbb{R}_{++}$, such that $p(0) = \limsup_{x \to 0^+} p(x)$ and $\liminf_{x \to 0^+} p(x) > 0$. Let $q_0 := \min\{x \in \mathbb{R}_{++} : p(x) = 0\}$ and $q_\mu : [0, q_0) \to \mathbb{R}$ be a function such that $q_\mu : z \mapsto \min \arg\max_{x \in [0, q_0]} p(z + x)\, x$. Let $(p, n, (K_1, .., K_n), (c_1, .., c_n)) \in O^p_{cvx}$. There exists a unique Cournot equilibrium for $(p, n, (K_1, .., K_n), (c_1, .., c_n))$ if:*

1. *$p(q + z)\, q$ is strictly concave in q over $(0, q_\mu(z))$, $\forall z \in [0, q_0)$,*
2. *$p(q + z)\, q$ and quasiconcave in q over $\mathbb{R}_+$,$\forall z \in \mathbb{R}_+$,*
3. *p is continuously differentiable over $(0, q_0)$,*
4. *for every integer $m \geq 1$, $\forall x \in (0, q_0)$,$\forall y \in [0, x]$,$\forall z \in (0, q_0 - x)$, $p'(x)\, y + mp(x) > 0$ entails that $p'(x)\, y + mp(x) > p'(x + z)(y + z) + mp(q + z)$.*

Proof. The reader may notice that $p(q_i + z)\, q_i - c_i(q_i)$ is strictly quasiconcave in q_i over K_i,$\forall z \in \mathbb{R}_+$:

i) if $z \in [0, q_0)$ it suffices to notice that the strict concavity of $p(q_i + z)\, q_i$ in q_i over $[0, q_\mu(z)]$ entails the strict concavity of $p(q_i + z)\, q_i - c_i(q_i)$ in q_i over $[0, q_\mu(z)] \cap K_i$ and that the nonincreasing monotonicity of $p(q_i + z)\, q_i$ in q_i over $[q_\mu(z), +\infty)$ entails the decreasing monotonicity of $p(q_i + z)\, q_i - c_i(q_i)$ in q_i over $[q_\mu(z), +\infty) \cap K_i$,

ii) if $z \geq q_0$ it suffices to notice that $p(q_i + z)\, q_i - c_i(q_i)$ is decreasing in q_i over K_i.

Let, $\forall i = 1, .., n$, $\mu_i \in \arg\max_{q \in K_i} p(q)\, q - c_i(q)$[31] and $\mu := \max\{\mu_1, .., \mu_n\}$. It can be easily noticed that $\mu < q_0$. It is immediate that if $n = 1$ then μ_1 is the unique Cournot equilibrium. Hence let us consider only the case $n > 1$.

It can be easily noticed that since $p(q_i + z)\, q_i - c_i(q_i) = 0$ when $q_i = 0$, whatever z be, then the Cournot equilibrium profits, if any, must be nonnegative. For this reason any vector $(\hat{q}_1, .., \hat{q}_n) \in \prod_{i=1}^n K_i$ such that $\sum_{i=1}^n \hat{q}_i \geq q_0$ cannot be a Cournot equilibrium: let $\hat{q} := \sum_{i=1}^n \hat{q}_i \geq q_0 > 0$, since for at least one firm, say j, $\hat{q}_j > 0$, then we must have that $p(\hat{q})\, \hat{q}_j - c_j(\hat{q}_j) = -c_j(\hat{q}_j) < 0$.

[31] Notice that there exists a unique maximizer.

The reader may notice that if $\mu = 0$ then $\mathbf{0} \in \mathbb{R}^n$ is a Cournot equilibrium for $(p, n, (K_1, .., K_n), (c_1, .., c_n))$: since $0 \in \underset{q \in K_i}{\arg\max} p(q_i + 0) q_i - c_i(q_i)$ $\forall i = 1, .., n$ then 0 maximizes the profit function of firm i when the overall production of other firms is 0, therefore $\mathbf{0}$ is a Cournot equilibrium.

Moreover if $\mu = 0$ $\mathbf{0} \in \mathbb{R}^n$ is the unique Cournot equilibrium. For the proof thereof simply notice that if $\mu = 0$ no firm can produce a positive quantity at any Cournot equilibrium: let us suppose that $(\hat{q}_1, ..., \hat{q}_n)$ is a Cournot equilibrium and $\hat{q}_j > 0$, since $p(q_j) q_j - c_j(q_j)$ is strictly quasiconcave in q_j and $0 \in \underset{q_j \in K_j}{\arg\max} p(q_j) q_j - c_j(q_j)$ we have that $0 > p(\hat{q}_j) \hat{q}_j - c_j(\hat{q}_j) \geq p(\hat{q}_1 + ... + \hat{q}_n) \hat{q}_j - c_j(\hat{q}_j)$ $\forall \hat{q}_j > 0$.

Let $\mu > 0$ and notice that any vector $(\hat{q}_1, .., \hat{q}_n) \in \prod_{i=1}^n K_i$ such that $\hat{q} := \sum_{i=1}^n \hat{q}_i < \mu$ cannot be a Cournot equilibrium: let us admit that $(\hat{q}_1, .., \hat{q}_n)$ is a Cournot equilibrium, let j be any firm such that $\mu_j = \mu > \hat{q}$, and let $\hat{q}_{-j} := \sum_{i \neq j} \hat{q}_i$, by definition $p(\mu)\mu - c_j(\mu) > p(\hat{q})\hat{q} - c_j(\hat{q})$, and therefore, since $-p(\hat{q})\hat{q}_{-j} \leq -p(\mu)\hat{q}_{-j}$, $p(\mu)(\mu - \hat{q}_{-j}) - c_j(\mu) > p(\hat{q})\hat{q}_j - c_j(\hat{q})$, by convexity of c_j we have $c_j(\mu) - c_j(\mu - \hat{q}_{-j}) \geq c_j(\hat{q}) - c_j(\hat{q}_j)$, therefore $p(\mu)(\mu - \hat{q}_{-j}) - c_j(\mu - \hat{q}_{-j}) > p(\hat{q})\hat{q}_j - c_j(\hat{q}_j)$, but this entails that $(\hat{q}_1, .., \hat{q}_n)$ cannot be Cournot equilibrium.

Let us show that if $\mu > 0$, $(\hat{q}_1, .., \hat{q}_n)$ is a Cournot equilibrium, $\max K_j$ exists and $\lim_{x \uparrow \max K_j} c'_{j-} = +\infty$[32] then $\hat{q}_j < \max K_j$: let $\hat{q}_{-j} := \sum_{i \neq j} \hat{q}_i$, we have showed above that $p(q_j + \hat{q}_{-j}) q_j - c_j(q_j)$ is strictly quasiconcave in q_j over K_j, therefore, if $\max K_j$ maximizes $p(q_j + \hat{q}_{-j}) q_j - c_j(q_j)$ in q_j over K_j then, we have that, $\forall q_j \in [0, \max K_j)$,

$$p(\max K_j + \hat{q}_{-j}) \max K_j - c_j(\max K_j) > p(q_j + \hat{q}_{-j}) q_j - c_j(q_j),$$

this entails that $c'_{j-}(\max K_j)$ exists; therefore if $\lim_{x \uparrow \max K_j} c'_{j-} = +\infty$ we must have that $\hat{q}_j < \max K_j$.

Let us show that when $\mu > 0$ two Cournot equilibria cannot exist. Let us admit that $(\hat{q}_1, .., \hat{q}_n)$ and $(\check{q}_1, .., \check{q}_n)$ are two different equilibria and that $0 < \hat{q}_1 + .. + \hat{q}_n = \check{q}_1 + .. + \check{q}_n = \bar{q} < q_0$, then there must exist a firm, say j, such that $\check{q}_j > \hat{q}_j \geq 0$. By the definition of a Cournot equilibrium and from the strict quasiconcavity of the profit function we must have that

$$p'(\bar{q}) \hat{q}_j + p(\bar{q}) - c'_{i+}(\hat{q}_j) \leq 0 \tag{1}$$

and that

$$p'(\bar{q}) \check{q}_j + p(\bar{q}) - c'_{i-}(\hat{q}_j) \geq 0, \tag{2}$$

since $p'(\bar{q}) \hat{q}_j + p(\bar{q}) - c'_{i+}(\hat{q}_j) \leq 0$, $p'(\bar{q})(\check{q}_j - \hat{q}_j) < 0$ and $c'_{i+}(\hat{q}_j) - c'_{i-}(\check{q}_j) \leq 0$ then

$$p'(\bar{q}) \check{q}_j + p(\bar{q}) - c'_{i-}(\check{q}_j) < 0, \tag{3}$$

[32] The reader should remember that $f'_-(x)$ denotes the left-hand derivative at $x \in A \subseteq \mathbb{R}$ (and $f'_+(x)$ denotes the right-hand derivative at x) of $f : A \rightarrow \mathbb{R}$.

a contradiction with (2). Let us now admit that $(\hat{q}_1, .., \hat{q}_n)$ and $(\check{q}_1, .., \check{q}_n)$ are two different equilibria and that $0 < \check{q}_1 + .. + \check{q}_n = \check{q} < \hat{q}_1 + .. + \hat{q}_n = \hat{q} < q_0$. Let $\Upsilon := \{i \in \{1, .., n\} : \check{q}_i < \hat{q}_i\}$, let $\hat{q}_\Upsilon := \sum_{i\in\Upsilon} \hat{q}_i$, and $\check{q}_\Upsilon := \sum_{i\in\Upsilon} \check{q}_i$, clearly $\hat{q}_\Upsilon - \check{q}_\Upsilon \geq \hat{q} - \check{q}$. Let $m := |\Upsilon| (\geq 1)$. Since $p'(\check{q})\check{q}_i + p(\check{q}) - c'_{i+}(\check{q}_i) \leq 0$ for every $i \in \Upsilon$ then

$$p'(\check{q})\check{q}_\Upsilon + mp(\check{q}) - \sum_{i\in\Upsilon} c'_{i+}(\check{q}_i) \leq 0 \tag{4}$$

and since $p'(\hat{q})\hat{q}_i + p(\hat{q}) - c'_{i-}(\hat{q}_i) \geq 0$ for every $i \in \Upsilon$ then

$$p'(\hat{q})\hat{q}_\Upsilon + mp(\hat{q}) - \sum_{i\in\Upsilon} c'_{i-}(\hat{q}_i) \geq 0. \tag{5}$$

Since $p'(\check{q})\check{q}_i + p(\check{q}) > 0$ for every $i \in \Upsilon$, then $p'(\check{q})\check{q}_\Upsilon + mp(\check{q}) > 0$. By assumption we must have:

$$p'(\check{q})\check{q}_\Upsilon + mp(\check{q}) > p'(\hat{q})(\hat{q} - \check{q} + \check{q}_\Upsilon) + mp(\hat{q}) \geq p'(\hat{q})(\hat{q}_\Upsilon) + mp(\hat{q}). \tag{6}$$

Since $\sum_{i\in\Upsilon} c'_{i+}(\check{q}_i) \leq \sum_{i\in\Upsilon} c'_{i-}(\hat{q}_i)$ we can conclude that

$$p'(\check{q})\check{q}_\Upsilon + mp(\check{q}) - \sum_{i\in\Upsilon} c'_{i+}(\check{q}_i) > p'(\hat{q})(\hat{q}_\Upsilon) + mp(\hat{q}) - \sum_{i\in\Upsilon} c'_{i-}(\hat{q}_i), \tag{7}$$

a contradiction with (4) and (5).

For every $i = 1, .., n$ let $\bar{c}'_{i-} : \mathring{K}_i \cup \{0\} \to \mathbb{R}$ be a function such that $\bar{c}'_{i-}(x) := c'_{i-}(x)\ \forall x \in \mathring{K}_i$ and $\bar{c}'_{i-}(0) := 0$, and

$$\bar{K}_i := \left[0, \max\{\, x \in K_i \cap [0, q_0] : \bar{c}'_{i-}(x) \leq p(0)\,\}\right].$$

Notice that $p'(x)\,0 + p(x) - \bar{c}'_{i-}(0) > 0\ \forall x \in [\mu, q_0)$ and that $p'(x)\,q_i + p(x) - \bar{c}'_{i-}(q_i)$ is decreasing and left-continuous in q_i over $K_i\ \forall x \in [\mu, q_0)$. Let, $\forall i = 1, .., n$,

$$b_i : [\mu, q_0) \to \bar{K}_i,\ b_i : x \mapsto \max\left\{q_i \in \bar{K}_i : p'(x)\,q_i + p(x) - \bar{c}'_{i-}(q_i) \geq 0\right\},$$

$$\text{and} \quad b : [\mu, q_0) \to \mathbb{R}, \quad b : x \mapsto \textstyle\sum_{i=1}^n b_i(x).$$

Let us show that there exists at least one value say $\bar{e}$ such that $b(\bar{e}) = \bar{e}$. First let us show that b_i is right-continuous over $[\mu, q_0)$; let $\bar{\delta} > 0$:

i) if $b_i(x) = \max \bar{K}_i$ then there exists a value $\varepsilon_1 > 0$ such that $b_i(x+\varepsilon) \leq b_i(x) + \bar{\delta}\ \forall \varepsilon \in (0, \varepsilon_1)$,

ii) if $b_i(x) < \max \bar{K}_i$ let $\delta_1 \in \left(0, \max \bar{K}_i - b_i(x)\right)$ be a value such that $\delta_1 \leq \bar{\delta}$, by continuity of p' and since $p'(x) < 0$ there exists a value ε_1 such that for every $\varepsilon \in (0, \varepsilon_1)$ $p'(x+\varepsilon)(b_i(x) + \delta_1) + p(x+\varepsilon) - \bar{c}'_{i-}(b_i(x) + \delta_1) < 0$ and therefore $b_i(x+\varepsilon) \leq b_i(x) + \delta_1 \leq b_i(x) + \bar{\delta}\ \forall \varepsilon \in (0, \varepsilon_1)$,

iii) if $b_i(x) = 0$ then there exists a value $\varepsilon_2 > 0$ such that $b_i(x+\varepsilon) \geq b_i(x) - \bar{\delta}$ $\forall \varepsilon \in (0, \varepsilon_2)$,

iv) if $b_i(x) > 0$ let $\delta_2 \in (0, b_i(x))$ be a value such that $\delta_2 \leq \bar{\delta}$, by continuity of p' and since $p'(x) < 0$ there exists a value ε_2 such that for every $\varepsilon \in (0, \varepsilon_2)$ $p'(x+\varepsilon)(b_i(x) - \delta_2) + p(x+\varepsilon) - \bar{c}'_{i-}(b_i(x) - \delta_2) > 0$ and therefore $b_i(x+\varepsilon) \geq b_i(x) - \delta_2 \geq b_i(x) - \bar{\delta}$ $\forall \varepsilon \in (0, \varepsilon_2)$,

now let $\bar{\varepsilon} := \min\{\varepsilon_1, \varepsilon_2\} > 0$, we can conclude that $\forall \bar{\delta} > 0$ there exists a value $\bar{\varepsilon} > 0$ such that $b_i(x) + \bar{\delta} \geq b_i(x+\varepsilon) \geq b_i(x) - \bar{\delta}$ $\forall \varepsilon \in (0, \bar{\varepsilon})$, which entails that b_i is right-continuous over $[\mu, q_0)$.

Similarly we can show that b_i is left-continuous over (μ, q_0). Hence b_i is continuous over $[\mu, q_0)$. Since there exists at least one firm, say j such that $\mu_j = \mu$, then $\mu = b_j(\mu)$. Hence $b(\mu) \geq \mu$. Since $p(q_0) = 0$ then $\lim_{x \to q_0} b_i(x) = \lim_{x \to q_0} b(x) = 0 < q_0$, $\forall i = 1, .., n$. Therefore there exists at least one value say $\bar{e}$ such that $b(\bar{e}) = \bar{e} \in [\mu, q_0)$.

Let us show that at any point $e \in [\mu, q_0)$ such that $b(e) = e$, $(b_1(e), ..., b_n(e))$ is a Cournot equilibrium. Let, $\forall i = 1, .., n$, $e_{-i} := \sum_{j \neq i} b_j(e)$. Since by assumption we have that $p(q_i + e_{-i}) q_i$ is quasiconcave in q_i over $\mathbb{R}_+$, $c_i(q_i)$ is increasing convex and continuous over K_i, and $p(q_i + e_{-i}) q_i$ is strictly concave over $(0, q_\mu(e_{-i}))$, then, $\forall i = 1, .., n$:

i) if $b_i(e) = 0$ then $p(e) - \bar{c}'_{i-}(0) \geq 0$ and $p(e) - c'_{i+}(0) \leq 0$, hence

$$b_i(e) \in \arg\max_{q_i \in K_i} p(q_i + e_{-i}) q_i - c_i(q_i),$$

ii) if $b_i(e) \in (0, \max \bar{K}_j)$ then $p'(e) b_i(e) + p(e) - \bar{c}'_{i-}(b_i(e)) = p'(e) b_i(e) + p(e) - c'_{i-}(b_i(e)) \geq 0$ and $p'(e) b_i(e) + p(e) - c'_{i+}(b_i(e)) \leq 0$, hence

$$b_i(e) \in \arg\max_{q_i \in K_i} p(q_i + e_{-i}) q_i - c_i(q_i),$$

iii) if $b_i(e) = \max \bar{K}_i > 0$ then $p'(e + q_i - \max \bar{K}_i) q_i + p(e + q - \max \bar{K}_i) - c'_{i+}(q_i) > 0$ $\forall q_i \in [0, \max \bar{K}_i)$,

iii)a) if $\max \bar{K}_i = \max K_i$ then

$$b_i(e) \in \arg\max_{q_i \in K_i} p(q_i + e_{-i}) q_i - c_i(q_i),$$

iii)b) if $\max \bar{K}_i < \max K_i$ then $p'(e) b_i(e) + p(e) - c'_{i+}(b_i(e)) \leq 0$, hence

$$b_i(e) \in \arg\max_{q_i \in K_i} p(q_i + e_{-i}) q_i - c_i(q_i).$$

We can conclude that, since $b_i(e) \in \arg\max_{q_i \in K_i} p(q_i + e_{-i}) q_i - c_i(q_i)$, $\forall i = 1, .., n$, then $(b_1(e), ..., b_n(e))$ is a Cournot equilibrium. Since two Cournot equilibria cannot exist $(b_1(e), ..., b_n(e))$ is the unique Cournot equilibrium. ■

Appendix D

Proposition 20 *Let $p : \mathbb{R}_+ \to \mathbb{R}_+$ be a price function, continuous over $\mathbb{R}_{++}$, null at some point of $\mathbb{R}_{++}$, such that $p(0) = \limsup_{x \to 0^+} p(x)$ and $\liminf_{x \to 0^+} p(x) > 0$. Let $q_0 := \min \{x \in \mathbb{R}_{++} : p(x) = 0\}$ and $q_\mu : [0, q_0) \to \mathbb{R}$, $q_\mu : z \mapsto \min \arg\max_{x \in [0, q_0]} p(z + x)\, x$. If p is continuously differentiable over $(0, q_0)$, $p(q)\, q$ is strictly concave in q over $(0, q_0)$ and $p(x) = 0\ \forall x \geq q_0$ then conditions 1 to 4 hold.*

Proof. Let us prove the validity of each condition:

1. Differentiability of p over $(0, q_0)$ is assumed.
2. Let us show that $p(q + z)\, q$ is strictly concave in q over $(0, q_\mu(z))$, $\forall z \in [0, q_0)$. When $z = 0$ the proof is immediate. Let $z \in (0, q_0)$, by the strict concavity of $p(q)\, q$ in q over $(0, q_0)$, $p'(q + z)(q + z) + p(q + z)$ is decreasing in q over $[0, q_0 - z)$. We have shown in a previous proposition that $p'(q + z) < 0\ \forall x \in [0, q_0 - z)$ and that p is decreasing over $[0, q_0]$. Since
$$p'(x_2 + z)(x_2 + z) + p(x_2 + z) < p'(x_1 + z)(x_1 + z) + p(x_1 + z), \quad (1)$$
$$\text{and} \quad p(x_2 + z) < p(x_1 + z), \quad (2)$$
whenever $0 \leq x_1 < x_2 < q_0 - z$, then we must have that:
-if $p'(x_2 + z) \leq p'(x_1 + z)$ then $p'(x_2 + z)\, x_2 < p'(x_1 + z)\, x_1$, by (2) we must have that
$$p'(x_2 + z)\, x_2 + p(x_2 + z) < p'(x_1 + z)\, x_1 + p(x_1 + z);$$
-if $p'(x_2 + z) > p'(x_1 + z)$ then $-p'(x_2 + z)\, z < -p'(x_1 + z)\, z$, by (1) we must have that
$$p'(x_2 + z)\, x_2 + p(x_2 + z) < p'(x_1 + z)\, x_1 + p(x_1 + z).$$
Therefore whenever $0 \leq x_1 < x_2 < q_0 - z$ we must have that
$$p'(x_2 + z)\, x_2 + p(x_2 + z) < p'(x_1 + z)\, x_1 + p(x_1 + z).$$
This entails that $p(q + z)\, q$ is strictly concave in q over $(0, q_0 - z)$, $\forall z \in [0, q_0)$. Since $q_0 - z > q_\mu(z)$ then $p(q + z)\, q$ is strictly concave in q over $(0, q_\mu(z))$, $\forall z \in [0, q_0)$.
3. In 2. we have shown that $p(q + z)\, q$ is strictly concave (and nonnegative) over $(0, q_0 - z)$ $\forall z \in [0, q_0)$, since $p(x + z) = 0\ \forall x \geq q_0 - z$ we have that $p(q + z)\, q$ is quasiconcave in q over $\mathbb{R}_+$, $\forall z \in \mathbb{R}_+$.

4. In 2. we have shown that $p(q+z)q$ is strictly concave over $(0, q_0 - z)$ $\forall z \in [0, q_0)$, hence $p'(x)y + p(x) > p'(x+z)(y+z) + p(x+z)$, $\forall x \in (0, q_0)$,$\forall y \in [0, x]$,$\forall z \in (0, q_0 - x)$; since p is nonincreasing over $\mathbb{R}_+$, we have that $p'(x)y + mp(x) > p'(x+z)(y+z) + mp(x+z)$ for every integer $m \geq 1$, $\forall x \in (0, q_0)$,$\forall y \in [0, x]$,$\forall z \in (0, q_0 - x)$.

This concludes the proof. ■

5 - Relevant Literature and Conclusions

There is a twofold difficulty in seeking necessary and sufficient conditions for the existence of a unique Cournot equilibrium. First, we must construct an economically significant problem. Then we must solve that problem. For a theoretical analysis a high degree of abstraction is desirable – the greater the abstraction the more general is the analysis – but that excessively general assumptions may yield cases which seem economically unreasonable, or at least unnatural. An example may clarify. Let us consider a duopoly, with price function[33] $p : \mathbb{R}_+ \to \mathbb{R}_+$, $p : x \mapsto 10$, consisting of two identical single-product firms with cost function $c : \mathbb{R}_+ \to \mathbb{R}_+$, $c : x \mapsto \begin{cases} 15x & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$. For this duopoly there exists a unique Cournot equilibrium[34]. But how sound and significant is this duopoly case? The example is clearly extreme and economically questionable, yet, without a proper formulation of the underlying economic problem, a proposition establishing necessary and sufficient conditions for the existence of a unique Cournot equilibrium might be compelled to contemplate cases as implausible as the previous one.

In the preceding sections we have specified the general conditions under which firms operate, and by relying on the separation between the cost function and the price function we have constructed the problem of the existence of a unique Cournot equilibrium. In such a construction we have imposed a desirable minimal degree of generality for the problem of the existence of a unique Cournot equilibrium, and, with the solution thereof, four necessary conditions has been established. Then it has been shown that these very conditions suffice for the existence of a unique Cournot equilibrium in a more general, and very significant, set of cases. Thus we have formulated, and demonstrated, two propositions on necessary and sufficient conditions for the existence of a unique Cournot equilibrium. It should be clear that our context and our construction, albeit justified, are not the only legitimate ones.

[33] The reader who has skipped Section 1 may simply replace the phrase "price function" with "inverse demand function".

[34] The Cournot equilibrium is $(1, 1)$.

This section deals with the literature concerning necessary and sufficient conditions for the existence of a unique Cournot equilibrium; a comparison with our analysis will be drawn by focussing on the differences in assumptions and in problem construction. Fortunately the literature on this matter is scanty and, to the writer's knowledge, consists of only two articles appeared in the *Review of Economic Studies* in 1987 and in 1991.

The first work, written by Charles D. Kolstad and Lars Mathiesen, establishes that when the profit functions of the firms of an oligopoly are twice continuously differentiable and pseudoconcave[35] with respect to own output, a *boundary condition*[36] and a *regularity condition*[37] are satisfied, then the positivity, for all Cournot equilibria, of the determinant of the opposite of the Jacobian of the vector function whose components are the marginal[38] profit functions of the firms with positive (equilibrium) output, is a necessary and sufficient conditions for the existence of a unique Cournot equilibrium.

It is worth noticing that in Kolstad and Mathiesen the productive capacity of each firm is the nonnegative real half-line and that the inverse demand function and the cost functions are twice continuously differentiable, and therefore we know nothing about firms with bounded productive capacity – a case we have regarded as extremely realistic – and we do not get any sound justification for twice continuous differentiability – which is assumed only for mathematical convenience – whereas, in our construction, we have proven the necessity of the continuous differentiability of the price function over the interior of its support. Again, and similarly, while they assume pseudoconcavity, with respect to own output, of the profit function of each firm, we have shown that a similar property is necessary. This point is particularly important since they provide no justification for this assumption, which, in itself, is not natural from an economic standpoint.

The second work, written by Gaudet and Salant, substantially follows the approach and the result of Kolstad and Mathiesen. Its major improvement is the extension of Kolstad and Mathiesen's result to degenerate equilibria, which are ruled out by Kolstad and Mathiesen's *regularity condition.* Gaudet and Salant argue that such a condition, apparently anodyne, may preclude the application of Kolstad and Mathiesen's result to the study of oligopolistic behavior, even in

[35] Let $A \subseteq \mathbb{R}^n$ be an open set and $f : A \to \mathbb{R}$ be differentiable. f is (strictly) pseudoconcave over A if, $\forall (\mathbf{x}, \mathbf{y}) \in A^2$ with $\mathbf{x} \neq \mathbf{y}$, $\langle \mathbf{y} - \mathbf{x}, \nabla f(\mathbf{x}) \rangle \leq 0$ implies $f(\mathbf{y})\,(<) \leq f(\mathbf{x})$. Notice that every pseudoconcave function is quasiconcave and that every strictly pseudoconcave function is strictly quasiconcave.

[36] Notice that pseudoconcavity alone is not sufficient to prove the existence of a Cournot equilibrium. The *boundary condition* ensures that an equilibrium exists and the set of equilibria lies on a compact and convex set.

[37] They define a regular Cournot market an oligopoly which satisfies two *regularity conditions.* According to the first *regularity condition* all Cournot equilibria are *non degenerate*, that is, if some firm i produces 0 at an equilibrium then the marginal cost of i at 0 is strictly greater than the price at that equilibrium. This very condition constitutes an assumption for their result.

According to the second *regularity condition* for each Cournot equilibrium the Jacobian of the marginal profits for the firms with positive output is nonsingular.

[38] Marginal with respect to own output.

very simple cases. However, notwithstanding this and other minor differences, the objections which can be raised against the structure of the problem devised by Kolstad and Mathiesen can be extended to that devised by Gaudet and Salant, since these latter's construction does not considerably differ from the former's one.

The weak spot of these constructions lies in having established the necessary and sufficient conditions for uniqueness of equilibrium on the very equilibrium points: as a result we do not know which properties must hold off the point of equilibrium. Moreover these conditions are established on the equilibrium properties of the profit functions, and these properties, undoubtedly, can be deduced only in connection with the set of Cournot equilibria of a specific oligopoly. Hence we have no clue about the role played by the oligopoly structure in determining the existence of a unique Cournot equilibrium, and therefore we are told nothing as to the extent to which the uniqueness of a Cournot equilibrium for a certain oligopoly occurs only "by accident".

In order to avoid accidental results we have considered classes of oligopolies instead of single oligopolies. Our result on the necessary conditions is formally stated by Proposition 5. This proposition, assuming essentially only continuity of the price functions, establishes necessary conditions on a price function for the existence of a unique Cournot equilibrium for every oligopoly with linear and increasing cost functions which can be constructed with that price function. Thus formulated the problem rests on the conceptual separation between the price function and the cost function, according to which, in principle, with the same price function can be associated many different oligopolies, all potentially possible. In that formulation we have considered set of oligopolies with linear cost functions because these are the most basic cost functions.

Our result on the sufficient conditions is formally stated by Proposition 17. It establishes that, when a price function[39] satisfies the necessary conditions of Proposition 5, every oligopoly with convex, increasing and continuous cost functions which can be constructed with that price function possesses a unique Cournot equilibrium. We have considered set of oligopolies with convex, increasing and continuous cost functions because, according to the context set forth in the first section, these are the most significant cost functions and because, as we have shown in the previous section, this set satisfies a useful closure property. Since the set of oligopolies considered in Proposition 17 is a superset of that considered in Proposition 5, it should be clear that both propositions provide necessary and sufficient conditions.

It may well happen that an oligopoly satisfies the conditions of Kolstad and Mathiesen, or those of Gaudet and Salant, and that one has no clue as to what variations in the number of firms or in their cost functions preserves the uniqueness of the Cournot equilibrium. In a problem of entry one may need to rely on conditions guaranteeing the uniqueness of the Cournot equilibrium of every possible oligopoly that can be generated by firms' entry decisions. The

[39] Also in this case the continuity assumptions on the price functions are the same as in Proposition 5.

results of Kolstad and Mathiesen and those of Gaudet and Salant, as such, do not guarantee anything with respect to sets of oligopolies resulting from entry decisions, simply for they consider only single oligopolies in and of themselves. On the contrary our result can be helpful in this respect: if the price function satisfies the conditions of Proposition 17 and if the post-entry cost functions of the potential entrants are convex increasing and continuous, then every possible oligopoly that can be generated by entry and exit decisions has a unique Cournot equilibrium.

Likewise, in many problems of horizontal mergers, one may need to rely on a proposition ensuring the uniqueness of the Cournot equilibrium of every possible oligopoly resulting from firms' merger decisions. Of course there does not exist a unique way of conceiving the effects of a merger: in the previous section we have introduced an assumption on the structure of the post-merger cost functions which is very natural from an economic viewpoint and we have shown that, under that assumption, the set of oligopolies considered in the formulation of Proposition 17 is closed under the operation of merger. Therefore Proposition 17 still holds for all the post-merger oligopolies which can be generated within the oligopolies considered in that proposition. On the contrary it well may happen that an oligopoly possesses a unique Cournot equilibrium, that it satisfies the conditions of Kolstad and Mathiesen, or those of Gaudet and Salant, and that a post-merger oligopoly does not possess a unique Cournot equilibrium, or that it possesses a unique equilibrium but it is not possible to use their theorem in the proof thereof. Once again this happens for they consider only single oligopolies in and of themselves.

In conclusion, we have deliberately constructed two results that, in an appropriate and significant context, provide necessary and sufficient conditions for the existence of a unique Cournot equilibrium. By construction our results may be easily and usefully employed in the analysis of problems concerning the investment decisions, the ownership structures and, broadly speaking, the production-related choices of oligopolistic firms. We have explained how and why the existing literature on necessary and sufficient conditions for the existence of a unique Cournot equilibrium may not be of use to these analyses. In this regard the results of the literature on merely sufficient conditions for the existence of a unique Cournot equilibrium turn out to be more useful. The reason for that lies in the separation, usual in this latter kind of literature, between conditions on price function and conditions on cost functions which allows to formulate the problem of uniqueness on classes of oligopolies rather than on single oligopolies. It is exactly on such a separation that our results have been developed. The wide literature on merely sufficient conditions is not reviewed here since the subject of this work is, strictly speaking, different and in the main more complex. Indeed, the sufficiency results of that literature are not in the least more general than ours either for being less general, as they very often are, or for being focussed on special oligopoly structures, such as duopolies and symmetric oligopolies. It goes without saying that the sufficient conditions which have not been proved to be necessary may always be expected to be generalizable.

Bibliography

[1] Amir,R. (1996). "Cournot Oligopoly and the Theory of Supermodular Games", *Games and Economic Behavior*, 15, 132-148.

[2] Aumann,R. and Brandeburger,A. (1995). "Epistemic Conditions for Nash Equilibrium", *Econometrica*, 63, 1161-1180.

[3] Bernheim,B.D. (1984). "Rationalizable Strategic Behavior", *Econometrica*, 52, 1007-1028.

[4] Bernheim,B.D. Peleg,B. and Whinston,M.D. (1987). " Coalition-Proof equilibria I. Concepts", *Journal of Economic Theory*, 42, 1-12.

[5] Bernheim,B.D. and Whinston,M.D. (1987). " Coalition-Proof equilibria II. Applications", *Journal of Economic Theory*, 42, 13-29.

[6] Bertrand,J. (1883). "Theorie Mathematique de la Richesse Sociale", *Journal des Savants*, 67, contained in "*Antoine-Augustine Cournot, Opere*". Milano Finanza.

[7] Border,K.C. (1985). "*Fixed Point Theorems with Applications to Economics and Game Theory*". Cambridge University Press. Cambridge, UK.

[8] Chamberlin,E.H. (1957). "On the Origin of Oligopoly", *The Economic Journal*, 67, 211-218.

[9] Cournot,A. (2006). "*Antoine-Augustine Cournot, Opere*". Milano Finanza.

[10] Cournot,A. (1838). "*Recherches sur les Principes Mathematiques de la Theorie des Richesses*". English translation by Bacon,N.T. (1897). "*Researches into the Mathematical Principles of the Theory of Wealth*".

[11] Edgeworth,F.Y. (1897). "La teoria pura del monopolio", *Giornale degli Economisti*, 40, 13-31.

[12] Farrel,J. and Shapiro,C. (1990). "Horizontal Mergers: An Equilibrium Analysis", *American Economic Review*, 80, 107-126.

[13] Frank,C.R.Jr. and Quandt,R.E. (1963). "On the Existence of Cournot Equilibrium", *International Economic Review*, 5, 92-100.

[14] Fudenberg,D. and Tirole,J. (1991). "*Game Theory*". The MIT Press, Cambridge, MA.

[15] Gabszewicz,J.J. and Vial,J.P. (1972). "Oligopoly "A la Cournot" in a General Equilibrium Analysis", *Journal of Economic Theory*, 4, 381-400.

[16] Gabszewicz,J.J. (2002). "*Strategic Multilateral Exchange: General Equilibrium with Imperfect Competition*". Edward Elgar, Chetenham, UK.

[17] Gaudet,G. and Salant,S.W. (1991). "Uniqueness of Cournot Equilibrium: New Results from Old Methods", *Review of Economic Studies*, 58, 399-404.

[18] Kolstad,C.D. and Mathiesen,L. (1987). "Necessary and Sufficient Conditions for Uniqueness of a Cournot Equilibrium", *Review of Economic Studies*, 54, 681- 690.

[19] Kreps,D.M. (1990). "*Game Theory and Economic Modelling*". Clarendon Press, Oxford.

[20] Kukushkin,N.S. (1994). "A Condition for the Existence of a Nash Equilibrium in Games with Public and Private Objectives", *Games and Economic Behavior*, 7, 177-192.

[21] Kukushkin,N.S. (1994). "A Fixed-Point Theorem for Decreasing Mappings", *Economics Letters*, 46, 23-26.

[22] Mas-Colell,A. Whinston,M. and Green,J. (1995). "*Microeconomic Theory*". Oxford University Press.

[23] Mas-Colell,A. (1980). "Noncooperative Approaches to the Theory of Perfect Competition: Presentation", *Journal of Economic Theory*, 22, 121-135.

[24] McManus,M. (1962). "Numbers and Size in Cournot Oligopoly", *Yorkshire Bulletin of Social and Economic Research*, 14, 14-22.

[25] McManus,M. (1964). "Equilibrium, Numbers and Size in Cournot Oligopoly", *Yorkshire Bulletin of Social and Economic Research*, 16, 68-75.

[26] Murphy,F.H. Sherali,H.D. and Soyster,A.L. (1982). "A Mathematical Programming Approach for Determining Oligopolistic Market Equilibrium", *Mathematical Programming*, 24, 92-106.

[27] Nash,J.F. (1950). "*Non-Cooperative Games*". Ph.D. Dissertation, Princeton University, Princeton, NJ.

[28] Nishimura,K. and Friedman,J. (1981). "Existence of Nash Equilibrium in n-Person Games Without Quasiconcavity", *International Economic Review*, 22, 637-648.

[29] Novshek,W. (1985). "On the Existence of a Cournot Equilibrium", *Review of Economic Studies*, 52, 85-98.

[30] Pareto,V. (1911). "*Economie Mathematiques*". Encyclopedie des Sciences Mathematiques, Book I, Vol. IV, Note 22. Contained in "*Antoine-Augustine Cournot, Opere*". Milano Finanza.

[31] Roberts,J. and Sonnenschein,H. (1976). "On the Existence of Cournot Equilibrium without Concave Profit Functions", *Journal of Economic Theory*, 13, 112-117.

[32] Roberts,J. and Sonnenschein,H. (1977). "On the Foundations of the Theory of Monopolistic Competition", *Econometrica*, 45, 101-113.

[33] Rosen,J.B. (1965). "Existence and Uniqueness of Equilibrium Points for Concave n-Person Games", *Econometrica*, 33, 520-534.

[34] Rudin,W. (1964). "*Principles of Mathematical Analysis*", 2nd Edition. McGraw-Hill.

[35] Saigal,R. and Simon,C. (1973). "Generic Properties of the Complementarity Problem", *Mathematical Programming*, 4, 324-335.

[36] Szidarovszky,F. and Yakowitz,S. (1977). "A New Proof of the Existence and Uniqueness of the Cournot Equilibrium", *International Economic Review*, 18, 787-789.

[37] Szidarovszky,F. and Yakowitz,S. (1982). "Contributions to Cournot Oligopoly Theory", *Journal of Economic Theory*, 28, 51-70.

[38] Vives,X. (1990). "Nash Equilibrium with Strategic Complementarities", *Journal of Mathematical Economic*, 19, 305-321.

[39] Walras,M.E.L. (1863). "*Les Principes de la Theorie des Richesses de M. Cournot*". Contained in "*Antoine-Augustine Cournot, Opere*". Milano Finanza.

[40] Watts,A. (2002). "Uniqueness of Equilibrium in Cost Sharing Games", *Journal of Mathematical Economics*, 37, 47-70.