UNIVERSITÀ
DI SIENA
1240

QUADERNI DEL DIPARTIMENTO
DI ECONOMIA POLITICA E STATISTICA

Marco P. Tucci

Robust control in the presence of a colored model error term in discrete time

n. 907 – Dicembre 2023

Robust control in the presence of a colored model error term in discrete time ⋆

Marco P. Tucci [a]

[a] *Piazza S. Francesco 7, 53100 Siena, Italy*

Abstract

By explicitly taking into account the welfare loss associated with the various controls, it has been recently shown that the usual, i.e. with white noise model error term, robust control framework in discrete time problems implies that both players (the controller and malevolent nature) optimize their objective functional by treating today's shock (either malevolent or not) as linearly uncorrelated to tomorrow's shock, see e.g. [38]. Therefore, by construction, the most common robust control framework implies that the game at time t is linearly uncorrelated with the game at time $t+1$. It is then useless to handle situations characterized by correlated malevolent shocks. In this paper it is assumed that the standard robust control problem in discrete time with unstructured uncertainty à la Hansen and Sargent, i.e. a nonparametric set of additive mean-distorting model perturbations, where the decision maker is assumed to be "probabilistically sophisticated" is characterized by a colored model error (or system disturbance) term. The new θ-constrained worst-case controls for the decision maker and malevolent nature are derived. The effect of this new assumption on the results of some well known models, like the permanent income model, is discussed.

Key words: Linear quadratic tracking problem; optimal control; robust optimization; distributional uncertainty; colored model error term.

1 Introduction

In the last three decades the study and impact of robust control techniques has spread to several fields outside the classical application domains in engineering. Robustness is now playing a significant role in the areas of economics,

⋆ M. P. Tucci. Tel. +390577232620. Fax +390577234618.
Email address: `tucci@unisi.it` (Marco P. Tucci).

network systems, biological systems and optimization. [1] A characteristic "feature of most robust control theory", observes [4, p. 19] , "is that the a priori information on the unknown model errors (or signals) is nonprobabilistic in nature, but rather is in terms of sets of possible realizations. Typically, though not always, the errors are bounded in some way. . . . As a consequence, robust control aims at synthesizing control mechanisms that control in a satisfactory fashion (e.g., stabilize, or bound, an output) a family of models." [2] Then "standard control theory tells a decision maker how to make optimal decisions when his model is correct (whereas) robust control theory tells him how to make good decisions when his model approximates a correct one" [12, p. 25]. In other words, by applying robust control the decision maker makes good decisions when it is statistically difficult to distinguish between his approximating model and the correct one using a time series of moderate size. "Such decisions are said to be robust to misspecification of the approximating model" [12, p. 27].

In most of the applications in macroeconomic the model error term, sometimes called system disturbance input in the engineering literature, is assumed independently, if not identically and independently, distributed. However, as recently shown in [38] the usual, i.e. with white noise model error term, robust control framework in discrete time problems implies that both players (the controller and malevolent nature) optimize their objective functional by treating today's shock (either malevolent or not) as linearly uncorrelated to tomorrow's shock. Therefore, by construction, the most common robust control framework implies that the game at time t is linearly uncorrelated with the game at time $t+1$. As pointed out in [27, p. 1325] "the system disturbance input . . . includes disturbances of various nature entering the system, such as white (or colored) noise, deterministic norm-bounded signals, and reference signals." The goal of the present work is to derive the robust control formulae in the special case of a colored model error (or system disturbance) term in discrete time. [3]

The remainder of the paper is organized as follows. Section 2 introduces the standard robust control problem, i.e. with white noise model error term, assuming unstructured uncertainty à la Hansen and Sargent, i.e. a nonparamet-

[1] See, e.g., [27] and the references therein cited.
[2] Some relevant references in economics include: [7], [8], [9], [10], [11], [12], [13], [15], [17], [18], [25], [29], [30], [31], [32], [34] and [33]. An early advocate of worst-case analysis in monetary policy design was [39]. However the use of the minimax approach in control theory goes back to the 60's as pointed out in [2, pp. 1-4]. See, e.g., [3] for a comprehensive treatment of the robust control methodology.
[3] See, e.g., [5] and [28] for useful surveys of the recent literature on distributional uncertainty, [6] and [26] for a discussion of alternative approaches to distributionally robust optimization.

ric set of additive mean-distorting model perturbations, where the decision maker is taken to be “probabilistically sophisticated.” The discussion is carried out in great detail with an eye to the use of this procedure in alternative set-ups. Section 3 considers the case of a colored model error term. An augmented state vector including both the original states and the colored error term is defined and the Bellman equation for the augmented system is introduced. Then the new θ-constrained worst-case controls for the decision maker and malevolent nature are derived following the same steps used in the standard case. Section 4 reports some numerical results. The effect of the new assumption on the results of some well known models, like the ‘robustized’ version of the MacRae problem described in [36], [37] and the permanent income model widely used in the robust control literature, is discussed. The main conclusions are summarized in Section 5.

2 Robust control à la Hansen and Sargent

This section heavily draws upon [12] for a presentation of the present approach. The focus is on a decision maker with complete state observation “who has a unique explicitly specified approximating model but concedes that the data might actually be generated by an unknown member of a set of models that surround the approximating model” [12, p. 140]. [4] Then the linear system

$$\boldsymbol{y}_{t+1} = \boldsymbol{A}\boldsymbol{y}_t + \boldsymbol{B}\boldsymbol{u}_t + \boldsymbol{C}\boldsymbol{\varepsilon}_{t+1} \quad \text{for} \quad t = 0, 1, ..., \infty \tag{1}$$

with $\boldsymbol{y}_t$ the $n \times 1$ vector of state variables at time t, $\boldsymbol{u}_t$ the $m \times 1$ vector of control variables and $\boldsymbol{\varepsilon}_{t+1}$ an $l \times 1$ identically and independently distributed (iid) Gaussian vector process with mean zero and an identity contemporaneous covariance matrix, is viewed as an approximation to the true unknown model

$$\boldsymbol{y}_{t+1} = \boldsymbol{A}\boldsymbol{y}_t + \boldsymbol{B}\boldsymbol{u}_t + \boldsymbol{C}(\boldsymbol{\varepsilon}_{t+1} + \boldsymbol{\omega}_{t+1}) \quad \text{for} \quad t = 0, ..., \infty. \tag{2}$$

The matrices of coefficients $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are assumed known and $\boldsymbol{y}_0$ given.

In Equation (1) the vector $\boldsymbol{\omega}_{t+1}$ denotes the “unknown” $l \times 1$ “process that can feed back in a possibly nonlinear way on the history of $\boldsymbol{y}_t$, (i.e.) $\boldsymbol{\omega}_{t+1} = \boldsymbol{g}_t(\boldsymbol{y}_t, \boldsymbol{y}_{t-1}, ...)$ where $\{\boldsymbol{g}_t\}$ is a sequence of measurable functions” [12, pp. 26-27]. It is introduced because the “iid random process . . . ($\boldsymbol{\varepsilon}_{t+1}$) can represent only a very limited class of approximation errors and in particular cannot depict such examples of misspecified dynamics as are represented in models with nonlinear and time-dependent feedback of $\boldsymbol{y}_{t+1}$ on past states” [12, p. 26]. When

[4] See [12, Ch. 2 and 7] for a complete discussion of robust control in the time domain and the whole set of assumptions. An alternative, shorter, introduction can be found in [13].

Equation (2) "generates the data it is as though the errors in ... (1) were conditionally distributed as $\mathcal{N}(\boldsymbol{\omega}_{t+1}, \boldsymbol{I}_l)$ rather than as $\mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_l)$"[12, p. 26]. To express the idea that (1) is a good approximation of (2) the $\boldsymbol{\omega}$'s are restrained by

$$E_0\left[\sum_{t=0}^{\infty}\beta^{t+1}\boldsymbol{\omega}'_{t+1}\boldsymbol{\omega}_{t+1}\right] \leq \eta_0 \quad \text{with} \quad 0 < \beta < 1 \tag{3}$$

where the symbol $'$ stands for transpose, E_0 denotes mathematical expectation evaluated with respect to model (2) and conditioned on $\boldsymbol{y}_0$ and η_0 measures the set of models surrounding the approximating model. [5]

"The decision makers distrust of his model ... (1) makes him want good decisions over a set of models ... (2) satisfying ... (3)" is written in [12, p. 27]. Then it is shown that this problem can be defined as a *multiplier robust control problem*

$$\max_{\boldsymbol{u}} \min_{\boldsymbol{\omega}} -E_0\left\{\sum_{t=0}^{\infty}\beta^t\left[r(\boldsymbol{y}_t, \boldsymbol{u}_t) - \theta\beta\boldsymbol{\omega}'_{t+1}\boldsymbol{\omega}_{t+1}\right]\right\}, \tag{4}$$

where $r(\boldsymbol{y}_t, \boldsymbol{u}_t)$ is the one-period loss function, subject to (2) with θ, $0 < \theta^* < \theta \leq \infty$, a penalty parameter restraining the minimizing choice of the $\{\boldsymbol{\omega}_{t+1}\}$ sequence.The problem can be reinterpreted as a two-player zero-sum game, the *multiplier game*, where one player is the decision maker maximizing the objective function by choosing the sequence for $\boldsymbol{u}$ and the other player is a malevolent nature choosing a feedback rule for a model-misspecification process $\boldsymbol{\omega}$ to minimize the same criterion function. [6] The "breakdown point" θ^* represents "a lower bound on θ^* that is required to keep the objective of the two-person zero-sum game convex in ... $(\boldsymbol{\omega}_{t+1})$ and concave in $\boldsymbol{u}_t$" [12, p. 161]. [7]

Therefore the robust rules for $\boldsymbol{u}_t$ and the worst-case shock $\boldsymbol{\omega}_{t+1}$ can be directly computed from the associated ordinary linear regulator problem. In particular, when the one-period loss function $r(\boldsymbol{y}_t, \boldsymbol{u}_t)$ is specified as

$$\boldsymbol{y}'_t\boldsymbol{Q}\boldsymbol{y}_t + 2\boldsymbol{y}'_t\boldsymbol{W}\boldsymbol{u}_t + \boldsymbol{u}'_t\boldsymbol{R}\boldsymbol{u}_t, \tag{5}$$

[5] See [12, p. 27].

[6] See [12, pp. 34-40].

[7] See [12, Ch. 7] for a further discussion of the restrictions on the robustness parameter θ. Alternatively the robust control problem can be formalized as the *constraint robust control problem*, or *constraint game*, defined as $\max_{\boldsymbol{u}} \min_{\boldsymbol{\omega}} - E_0\left[\sum_{t=0}^{\infty}\beta^t r(\boldsymbol{y}_t, \boldsymbol{u}_t)\right]$ subject to (2)-(3) where $\eta^* > \eta_0$ and η^* "measures the largest set of perturbations against which it is possible to seek robustness" [12, pp. 32-34]. If the parameters η_0 and θ are appropriately related the multiplier and constraint problems, or games, have equivalent outcomes in the sense that if there exists a solution $\boldsymbol{u}^*$, $\boldsymbol{\omega}^*$ to the former, the same $\boldsymbol{u}^*$ solves also the latter problem with the largest set of alternative models, i.e. $\eta_0 = \eta_0^* = E_0\left[\sum_{t=0}^{\infty}\beta^t(\boldsymbol{\omega}^*_{t+1})'\boldsymbol{\omega}^*_{t+1}\right]$[12, pp. 159-160].

with $\boldsymbol{Q}$ a positive semi-definite matrix, $\boldsymbol{R}$ a positive definite matrix, $\boldsymbol{W}$ an $n \times m$ array,, the robust control rule is derived by extremizing, i.e. maximizing with respect to $\boldsymbol{u}_t$ and minimizing with respect to $\boldsymbol{\omega}_{t+1}$, the objective function [8]

$$-E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[r(\boldsymbol{y}_t, \tilde{\boldsymbol{u}}_t) \right] \right\}, \tag{6}$$

with

$$r(\boldsymbol{y}_t, \tilde{\boldsymbol{u}}_t) = \boldsymbol{y}_t' \boldsymbol{Q} \boldsymbol{y}_t + 2\boldsymbol{y}_t' \tilde{\boldsymbol{W}} \tilde{\boldsymbol{u}}_t + \tilde{\boldsymbol{u}}_t' \tilde{\boldsymbol{R}} \tilde{\boldsymbol{u}}_t \tag{7}$$

subject to

$$\boldsymbol{y}_{t+1} = \boldsymbol{A}\boldsymbol{y}_t + \tilde{\boldsymbol{B}} \tilde{\boldsymbol{u}}_t + \boldsymbol{C}\boldsymbol{\varepsilon}_{t+1} \quad \text{for} \quad t = 0, \ldots, \infty \tag{8}$$

where [9]

$$\tilde{\boldsymbol{R}} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{O} \\ \boldsymbol{O} & -\theta\beta \boldsymbol{I}_l \end{bmatrix}, \tilde{\boldsymbol{u}}_t = \begin{bmatrix} \boldsymbol{u}_t \\ \boldsymbol{\omega}_{t+1} \end{bmatrix}, \tilde{\boldsymbol{B}} = [\boldsymbol{B}\ \boldsymbol{C}] \tag{9}$$

and $\tilde{\boldsymbol{W}} = [\boldsymbol{W}\ \boldsymbol{O}]$ with the $\boldsymbol{O}$'s denoting null arrays of appropriate dimension. [10]

Setting $\boldsymbol{\varepsilon}_{t+1} = \mathbf{0}$ and writing the optimal value of (6) as $-\boldsymbol{y}_t' \boldsymbol{P}_t \boldsymbol{y}_t$, [11] the Bellman equation looks like [12]

$$\begin{aligned} -\boldsymbol{y}_t' \boldsymbol{P}_t \boldsymbol{y}_t = \underset{\tilde{\boldsymbol{u}}}{ext} - \Big[& \boldsymbol{y}_t' \boldsymbol{Q}_t \boldsymbol{y}_t + \boldsymbol{u}_t' \boldsymbol{R}_t \boldsymbol{u}_t - \theta\beta \boldsymbol{\omega}_{t+1}' \boldsymbol{\omega}_{t+1} \\ & + 2\boldsymbol{y}_t' \boldsymbol{W}_t \boldsymbol{u}_t + \boldsymbol{y}_{t+1}' \boldsymbol{P}_{t+1} \boldsymbol{y}_{t+1} \Big] \end{aligned} \tag{10}$$

with $\boldsymbol{P}_{t+1} = \beta \boldsymbol{P}_t$, $\boldsymbol{Q}_{t+1} = \beta \boldsymbol{Q}_t$, $\boldsymbol{W}_{t+1} = \beta \boldsymbol{W}_t$ and $\boldsymbol{R}_{t+1} = \beta \boldsymbol{R}_t$. Then expressing the right-hand side of (10) only in terms of $\boldsymbol{y}_t$ and $\tilde{\boldsymbol{u}}_t$ and extremizing it yields the optimal control for the decision maker

$$\begin{aligned} \boldsymbol{u}_t = & - \left(\boldsymbol{R}_t + \boldsymbol{B}' \boldsymbol{P}_{t+1} \boldsymbol{B} \right)^{-1} \\ & \times \left[\left(\boldsymbol{B}' \boldsymbol{P}_{t+1} \boldsymbol{A} + \boldsymbol{W}_t' \right) \boldsymbol{y}_t + \boldsymbol{B}' \boldsymbol{P}_{t+1} \boldsymbol{B} \boldsymbol{\omega}_{t+1} \right] \end{aligned} \tag{11}$$

---

[8] Indeed, as pointed out in [21, p. 342, footnote 4] Equation (6) is a functional dependent on the control trajectory whereas the solution to the problem is a function dependent on the parameters given by the initial state $\boldsymbol{y}_0$ and the initial time $t = 0$.

[9] The penalty matrix $\tilde{\boldsymbol{R}}$ implies that each component of the vector process $\boldsymbol{\omega}_{t+1}$is penalized in the same way.

[10] The one-period loss function in Equation (5) implies that the desired path for $\boldsymbol{y}$ and $\boldsymbol{u}$ are 0. See, e.g., [37].

[11] Using the deterministic counterpart to (6) and (8) allows to simplify some formulas by dropping constants from the value function without affecting the formulas for the decision rules. See, e.g., [12, p. 33].

[12] The constant term appearing on the right-hand side and on the left-hand side of the equation have been dropped because they do not affect the solution of the optimization problem. See, e.g., Equation (2.5.3) in [12, Ch. 2].

and the optimal control for the malevolent nature

$$\boldsymbol{\omega}_{t+1} = \left(\theta\beta \boldsymbol{I}_l - \boldsymbol{C}'\boldsymbol{P}_{t+1}\boldsymbol{C}\right)^{-1} \boldsymbol{C}'\boldsymbol{P}_{t+1} \times \left(\boldsymbol{A}\boldsymbol{y}_t + \boldsymbol{B}\boldsymbol{u}_t\right). \tag{12}$$

It follows that the θ-constrained worst-case controls are [13]

$$\boldsymbol{u}_t = -\boldsymbol{\Delta}_{t+1}^{-1}\left(\boldsymbol{B}'\boldsymbol{P}_{t+1}^{*}\boldsymbol{A} + \boldsymbol{W}_t'\right)\boldsymbol{y}_t = \boldsymbol{G}_{t+1}\boldsymbol{y}_t \tag{13}$$

where $\boldsymbol{\Delta}_{t+1} = \boldsymbol{R} + \boldsymbol{B}'\boldsymbol{P}_{t+1}^{*}\boldsymbol{B}$, $\boldsymbol{G}_{t+1}$ is implicitely defined and [14]

$$\boldsymbol{\omega}_{t+1} = \boldsymbol{\Theta}_{t+1}^{-1}\boldsymbol{C}'\boldsymbol{P}_{t+1}\left(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{G}_{t+1}\right)\boldsymbol{y}_t = \boldsymbol{H}_{t+1}\boldsymbol{y}_t \tag{14}$$

with $\boldsymbol{\Theta}_{t+1} = \theta\beta\boldsymbol{I}_l - \boldsymbol{C}'\boldsymbol{P}_{t+1}\boldsymbol{C}$, $\boldsymbol{H}_{t+1}$ implicitely defined and [15]

$$\boldsymbol{P}_{t+1}^{*} = \boldsymbol{P}_{t+1} + \boldsymbol{P}_{t+1}\boldsymbol{C}\boldsymbol{\Theta}_{t+1}^{-1}\boldsymbol{C}'\boldsymbol{P}_{t+1}. \tag{15}$$

The "robust" Riccati matrix $\boldsymbol{P}_{t+1}^{*}$ is always greater or equal $\boldsymbol{P}_{t+1}$ because it is assumed that, in the "admissible" region, the parameter θ is large enough to make $(\theta\beta\boldsymbol{I}_l - \boldsymbol{C}'\boldsymbol{P}_{t+1}\boldsymbol{C})$ positive definite. [16] The two Riccati matrices are equal when $\theta = \infty$. [17]

As noted in [12, p. 11], the first order conditions for problem (10) subject to (8) imply the matrix Riccati equation

$$\boldsymbol{P}_t = \boldsymbol{Q}_t + \boldsymbol{A}'\boldsymbol{P}_{t+1}\boldsymbol{A} - \left(\boldsymbol{A}'\boldsymbol{P}_{t+1}\tilde{\boldsymbol{B}} + \tilde{\boldsymbol{W}}_t\right) \times \left(\tilde{\boldsymbol{R}}_t + \tilde{\boldsymbol{B}}'\boldsymbol{P}_{t+1}\tilde{\boldsymbol{B}}\right)^{-1}\left(\boldsymbol{A}'\boldsymbol{P}_{t+1}\tilde{\boldsymbol{B}} + \tilde{\boldsymbol{W}}_t\right)' \tag{16}$$

[13] See, e.g., equations (7.C.18)-(7.C.19) in [12, p. 169]. As suggested on page 35 of the same reference, equation (10) "can be represented as" $-\boldsymbol{y}_t'\boldsymbol{P}_t\boldsymbol{y}_t = \max_{\boldsymbol{u}} -\left[\boldsymbol{y}_t'\boldsymbol{Q}_t\boldsymbol{y}_t + \boldsymbol{u}_t'\boldsymbol{R}_t\boldsymbol{u}_t + 2\boldsymbol{y}_t'\boldsymbol{W}_t\boldsymbol{u}_t + \boldsymbol{y}_{t+1}'\boldsymbol{P}_{t+1}\boldsymbol{y}_{t+1}\right]$ subject to the approximating model (1) instead of the distorted model (2).

[14] See, e.g., Equation (7.C.9) in [12, p. 168]. As pointed out on page 139 of the same work, the "two-player zero-sum dynamic games ... (i.e.) an effectively static Stackelberg multiplier game in which a ... player at time 0 chooses a history-dependent sequence of controls ... (and) a Markov perfect multiplier game in which both players choose sequentially ... have identical outcomes" both when the $\boldsymbol{\omega}$-player chooses before the $\boldsymbol{u}$-player, at time 0 or in each period $t \geq 0$, or vice versa.

[15] See, e.g., equations (2.5.6) on p. 35 and (7.C.10) on p. 168 in [12] where the quantity $\beta^{-1}\boldsymbol{P}_{t+1}^{*}$ is denoted by $\mathcal{D}(\boldsymbol{P})$.

[16] See, e.g., Theorem 7.6.1 (assumption v) in [12, p. 150].

[17] The parameter θ is closely related to the risk-sensitivity parameter, say σ, appearing in intertemporal preferences obtained recursively. Namely, it can be interpreted as minus the inverse of σ. See, e.g., [12, pp. 40-41, 45 and 225-236], [17] and the references therein cited.

where $\tilde{\boldsymbol{R}}_t = \beta^t \tilde{\boldsymbol{R}}$ and $\tilde{\boldsymbol{W}}_t = \beta^t \tilde{\boldsymbol{W}}$. It is straightforward to show that the right-hand side of (16) can be rewritten as

$$\begin{aligned} &\boldsymbol{Q}_t + \boldsymbol{A}'\boldsymbol{P}^*_{t+1}\boldsymbol{A} - \left(\boldsymbol{A}'\boldsymbol{P}^*_{t+1}\boldsymbol{B} + \boldsymbol{W}_t\right) \\ &\times \boldsymbol{\Delta}^{-1}_{t+1}\left(\boldsymbol{A}'\boldsymbol{P}^*_{t+1}\boldsymbol{B} + \boldsymbol{W}_t\right)' \end{aligned}$$

with $\boldsymbol{\Delta}_{t+1}$ defined as above. Equation (16) reduces to the usual Riccati recursion of the linear quadratic tracking control problem when $\boldsymbol{\omega}_{t+1} = \mathbf{0}$.

It follows that under this set of assumptions, namely all the elements of arrays $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are known and constant, $\boldsymbol{y}_0$ is given and the model error term is white noise [18] the sequence of θ-constrained optimal controls for the decision maker and the malevolent nature defined in (13 and (14), respectively, are uncorrelated. Namely,

$$\begin{aligned} cov\left(\boldsymbol{u}_t, \boldsymbol{u}_{t+1}\right) &= cov\left(\boldsymbol{G}_{t+1}\boldsymbol{y}_t, \boldsymbol{G}_{t+2}\boldsymbol{y}_{t+1}\right) \\ &= \boldsymbol{G}_{t+1}E_0\left(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}'_{t+1}\right)\boldsymbol{G}'_{t+2} = \boldsymbol{O} \end{aligned}$$

and

$$\begin{aligned} cov\left(\boldsymbol{\omega}_{t+1}, \boldsymbol{\omega}_{t+2}\right) &= cov\left(\boldsymbol{H}_{t+1}\boldsymbol{y}_t, \boldsymbol{H}_{t+2}\boldsymbol{y}_{t+1}\right) \\ &= \boldsymbol{H}_{t+1}E_0\left(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}'_{t+1}\right)\boldsymbol{H}'_{t+2} = \boldsymbol{O} \end{aligned}$$

when the $\boldsymbol{G}$'s and $\boldsymbol{H}$'s are defined as above. This implies that both players (the controller and malevolent nature) optimize their objective functional by treating today's shock (either malevolent or not) as linearly uncorrelated to tomorrow's shock. [19] By using the results reported in [38] these conclusions can be extended also to situations where there are multiple penalty functions (i.e. more than one θ), in other words cases where the decision maker is not "probabilistically sophisticated" [12, p. 383]. This occurs when the decision maker does not observe parts of the state useful to forecast relevant variables. Then the approximating model includes an ordinary (i.e. non robust) Kalman filter estimator of this hidden portion of the state. This is sometimes referred to as the "robust filtering without commitment" problem. Analogously, these results extend to cases where robust control is applied to situations where uncertainty is related to unknown structural parameters as in [7] and [8].

[18] Recently robust control in continuous time models as gained momentum in Economics. The model error term in (1) is typically assumed a multivariate Brownian motion and robustness, i.e. the worst-case probability, is represented by a drift distortion that nature chooses to minimize the objective function. In this type of models perturbations to the approximating model are generally represented in terms of martingales. See, e.g., [14], [15], [19] and [16]. A closer look at this strand of the literature goes beyond the scope of the present work.
[19] See, e.g., [38].

3 Robust control in the presence of a colored model error term

When the model error term $\boldsymbol{\varepsilon}$ in (2) is colored noise it can be modeled as

$$\boldsymbol{\varepsilon}_{t+1} = \boldsymbol{\Phi}\boldsymbol{\varepsilon}_t + \boldsymbol{D}\boldsymbol{\xi}_{t+1} \quad \text{for} \quad t = 0, ..., \infty \tag{17}$$

with $\boldsymbol{\xi}_{t+1}$ an $l \times 1$ *iid* Gaussian vector process with mean zero and an identity contemporaneous covariance matrix, and the other arrays in (17) appropriately defined.[20] By defining an augmented state vector $\tilde{\boldsymbol{y}}_t$ having dimension $n + l$, i.e. the number of original states plus the number of colored error components, the problem can be cast as in the previous section and the solution readily obtained. Namely, the robust control rule is derived by extremizing, i.e. maximizing with respect to $\boldsymbol{u}_t$ and minimizing with respect to $\boldsymbol{\omega}_{t+1}$ the objective function

$$-E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[r(\tilde{\boldsymbol{y}}_t, \tilde{\boldsymbol{u}}_t) \right] \right\}, \tag{18}$$

with the one-period loss function specified as

$$r(\tilde{\boldsymbol{y}}_t, \tilde{\boldsymbol{u}}_t) = \tilde{\boldsymbol{y}}_t' \tilde{\boldsymbol{Q}} \boldsymbol{y}_t + 2\tilde{\boldsymbol{y}}_t' \tilde{\boldsymbol{W}} \tilde{\boldsymbol{u}}_t + \tilde{\boldsymbol{u}}_t' \tilde{\boldsymbol{R}} \tilde{\boldsymbol{u}}_t \tag{19}$$

where

$$\tilde{\boldsymbol{y}}_t = \begin{bmatrix} \boldsymbol{y}_t \\ \boldsymbol{\varepsilon}_{t+1} \end{bmatrix}, \tag{20}$$

$\tilde{\boldsymbol{u}}_t$ and $\tilde{\boldsymbol{R}}$ are defined as above, $\tilde{\boldsymbol{Q}}_t$ is block diagonal with $diag(\tilde{\boldsymbol{Q}}_t) = diag(\boldsymbol{Q}, \boldsymbol{O})$ and $diag(\tilde{\boldsymbol{W}}_t) = diag(\boldsymbol{W}, \boldsymbol{O})$ with the $\boldsymbol{O}$'s denoting null arrays of appropriate dimension. The extremization of Equation (18) is subject to

$$\tilde{\boldsymbol{y}}_{t+1} = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{y}}_t + \tilde{\boldsymbol{B}}\tilde{\boldsymbol{u}}_t + \tilde{\boldsymbol{D}}\boldsymbol{\xi}_{t+1} \quad \text{for} \quad t = 0, \ldots, \infty \tag{21}$$

where

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{O} & \boldsymbol{\Phi} \end{bmatrix}, \tilde{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix} \text{ and } \tilde{\boldsymbol{D}} = \begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{D} \end{bmatrix}. \tag{22}$$

Setting $\boldsymbol{\xi}_{t+1} = \boldsymbol{0}$ and writing the optimal value of (18) as $-E_t\left(\tilde{\boldsymbol{y}}_t' \tilde{\boldsymbol{P}}_t \boldsymbol{y}_t\right)$, the Bellman equation for the colored error model case looks like

$$\begin{aligned} -E_t\left(\tilde{\boldsymbol{y}}_t' \tilde{\boldsymbol{P}}_t \tilde{\boldsymbol{y}}_t\right) = \underset{\tilde{\boldsymbol{u}}}{ext} - E_t \Big[& \tilde{\boldsymbol{y}}_t' \tilde{\boldsymbol{Q}}_t \boldsymbol{y}_t + \tilde{\boldsymbol{u}}_t' \tilde{\boldsymbol{R}}_t \tilde{\boldsymbol{u}}_t \\ & + 2\tilde{\boldsymbol{y}}_t' \tilde{\boldsymbol{W}}_t \tilde{\boldsymbol{u}}_t + \tilde{\boldsymbol{y}}_{t+1}' \tilde{\boldsymbol{P}}_{t+1} \tilde{\boldsymbol{y}}_{t+1} \Big] \end{aligned} \tag{23}$$

[20] In general $\boldsymbol{\varepsilon}_{t+1}$ is the outcome of the system $\boldsymbol{\varepsilon}_{t+1} = \boldsymbol{C}_2 \boldsymbol{\zeta}_t$ with $\boldsymbol{\zeta}_{t+1} = \boldsymbol{\Phi}\boldsymbol{\zeta}_t + \boldsymbol{D}\boldsymbol{\xi}_{t+1}$ where $\boldsymbol{C}_2$ is an $l \times s$ matrix, $\boldsymbol{\zeta}_t$ and $\boldsymbol{\xi}_{t+1}$ are vectors of dimension $s \times 1$ and the other arrays are appropriately defined. See, e.g., [1, p. 299]. It should be noticed that any ARMA model can be rewritten as a multivariate AR(1) process [20, p. 103].

where the desired paths for the states and controls are set to 0 to save on notation and $\tilde{\boldsymbol{P}}_{t+1} = \beta\tilde{\boldsymbol{P}}_t$, $\tilde{\boldsymbol{Q}}_{t+1} = \beta\tilde{\boldsymbol{Q}}_t$, $\tilde{\boldsymbol{W}}_{t+1} = \beta\tilde{\boldsymbol{W}}_t$ and $\tilde{\boldsymbol{R}}_{t+1} = \beta\tilde{\boldsymbol{R}}_t$. Then substituting the system equations into the optimal cost expression, applying the expectation operator and using the fact that, when $\tilde{\boldsymbol{P}}_{t+1}$ is known,[21]

$$E_t\left(\tilde{\boldsymbol{y}}_t'\tilde{\boldsymbol{P}}_t\tilde{\boldsymbol{y}}_t\right) = \bar{\boldsymbol{y}}_t'\tilde{\boldsymbol{P}}_t\bar{\boldsymbol{y}}_t + \mathrm{Tr}\left[\tilde{\boldsymbol{P}}_t\boldsymbol{\Sigma}_t\right] \tag{24}$$

with Tr denoting the Trace operator,

$$\begin{aligned}\bar{\boldsymbol{y}}_t &= E_t\begin{bmatrix}\boldsymbol{y}_t\\ \boldsymbol{\varepsilon}_{t+1}\end{bmatrix} = \begin{bmatrix}\boldsymbol{y}_t\\ E_t\left(\boldsymbol{\varepsilon}_{t+1}\right)\end{bmatrix},\\ \tilde{\boldsymbol{P}}_t &= \begin{bmatrix}\underset{n\times n}{\boldsymbol{P}_{11}} & \boldsymbol{P}_{12}\\ \boldsymbol{P}_{21} & \underset{l\times l}{\boldsymbol{P}_{22}}\end{bmatrix}_t\end{aligned} \tag{25}$$

the problem is solved. The matrix $\boldsymbol{\Sigma}_t$ denotes the $cov(\tilde{\boldsymbol{y}}_t, \tilde{\boldsymbol{y}}_t)$ and is block diagonal with $diag(\boldsymbol{\Sigma}_t) = diag(\boldsymbol{O}, \boldsymbol{D}\boldsymbol{D}')$ where $n \times n$ and $l \times l$ are, respectively, the dimensions of the two diagonal blocks. Finally, t should be stressed that the North-West block of the augmented Riccati matrix $\tilde{\boldsymbol{P}}_t$, namely $\boldsymbol{P}_{11,t}$ is identical to the array $\boldsymbol{P}_t$ in (16).

The feedback rule is obtained from the first order conditions of the extremization. Rewriting Equation (23) in terms of the original arrays and assuming that all the coefficients in $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$ and $\boldsymbol{\Phi}$ are known, it can be shown that the robust control for the decision maker in the presence of colored noise is

$$\begin{aligned}\boldsymbol{u}_t^c = -\boldsymbol{\Delta}_{11,t+1}^{-1}\Big[&\left(\boldsymbol{B}'\boldsymbol{P}_{11,t+1}^*\boldsymbol{A} + \boldsymbol{W}_t'\right)\boldsymbol{y}_t\\ &+\boldsymbol{B}'\boldsymbol{P}_{11,t+1}^*\boldsymbol{C}_{t+1}^*E_t\left(\boldsymbol{\varepsilon}_{t+1}\right)\Big]\end{aligned} \tag{26}$$

with $\boldsymbol{P}_{11,t+1} \equiv \boldsymbol{P}_{t+1}$, $\boldsymbol{\Delta}_{11,t+1} = \boldsymbol{R} + \boldsymbol{B}'\boldsymbol{P}_{11,t+1}^*\boldsymbol{B} \equiv \boldsymbol{\Delta}_{t+1}$, $\boldsymbol{C}_{t+1}^* = \boldsymbol{C} + \left(\boldsymbol{P}_{11,t+1}^*\right)^{-1}\boldsymbol{P}_{12,t+1}^*\boldsymbol{\Phi}$ and the associated malevolent nature shock is

$$\begin{aligned}\boldsymbol{\omega}_{t+1}^c = \;&\boldsymbol{\Theta}_{11,t+1}^{-1}\boldsymbol{C}'\boldsymbol{P}_{11,t+1}\left(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{u}_t^c\right)\\ &+\boldsymbol{\Theta}_{11,t+1}^{-1}\boldsymbol{C}'\boldsymbol{P}_{11,t+1}\boldsymbol{C}_{t+1}^cE_t\left(\boldsymbol{\varepsilon}_{t+1}\right)\end{aligned} \tag{27}$$

with $\boldsymbol{\Theta}_{11,t+1} = \theta\beta\boldsymbol{I}_l - \boldsymbol{C}'\boldsymbol{P}_{11,t+1}\boldsymbol{C} \equiv \boldsymbol{\Theta}_{t+1}$ and $\boldsymbol{C}_{t+1}^c = \boldsymbol{C} + \left(\boldsymbol{P}_{11,t+1}\right)^{-1}\boldsymbol{P}_{12,t+1}\boldsymbol{\Phi}$.

The first order conditions for problem (23) subject to (21) imply the matrix

[21] See, e.g., [22, Appendix B].

Riccati equation

$$\begin{aligned}\tilde{\boldsymbol{P}}_t = \tilde{\boldsymbol{Q}}_t + \tilde{\boldsymbol{A}}'\tilde{\boldsymbol{P}}_{t+1}\tilde{\boldsymbol{A}} - \left(\tilde{\boldsymbol{A}}'\tilde{\boldsymbol{P}}_{t+1}\tilde{\boldsymbol{B}} + \tilde{\boldsymbol{W}}_t\right) \\ \times \left(\tilde{\boldsymbol{R}}_t + \tilde{\boldsymbol{B}}'\tilde{\boldsymbol{P}}_{t+1}\tilde{\boldsymbol{B}}\right)^{-1} \left(\tilde{\boldsymbol{A}}'\tilde{\boldsymbol{P}}_{t+1}\tilde{\boldsymbol{B}} + \tilde{\boldsymbol{W}}_t\right)'.\end{aligned} \tag{28}$$

Then the recursions for the different blocks in terms of the original arrays look like

$$\begin{aligned}\boldsymbol{P}_{11,t} = \boldsymbol{Q}_t + \boldsymbol{A}'\boldsymbol{P}^*_{11,t+1}\boldsymbol{A} - \left(\boldsymbol{A}'\boldsymbol{P}^*_{11,t+1}\boldsymbol{B} + \boldsymbol{W}_t\right) \\ \times \boldsymbol{\Delta}^{-1}_{11,t+1} \left(\boldsymbol{A}'\boldsymbol{P}^*_{11,t+1}\boldsymbol{B} + \boldsymbol{W}_t\right)'.\end{aligned} \tag{29}$$

$$\begin{aligned}\boldsymbol{P}_{12,t} &= \boldsymbol{A}'\boldsymbol{P}^*_{11,t+1}\boldsymbol{C} + \boldsymbol{A}'\boldsymbol{P}^*_{12,t+1}\boldsymbol{\Phi} \\ &- \left(\boldsymbol{A}'\boldsymbol{P}^*_{11,t+1}\boldsymbol{B} + \boldsymbol{W}_t\right) \\ &\times \boldsymbol{\Delta}^{-1}_{11,t+1}\boldsymbol{B}' \left(\boldsymbol{P}^*_{11,t+1}\boldsymbol{C} + \boldsymbol{P}^*_{12,t+1}\boldsymbol{\Phi}\right).\end{aligned} \tag{30}$$

$$\begin{aligned}\boldsymbol{P}_{22,t} &= \boldsymbol{C}'\boldsymbol{P}^*_{11,t+1}\boldsymbol{C} + \boldsymbol{\Phi}'\boldsymbol{P}^*_{21,t+1}\boldsymbol{C} \\ &+ \boldsymbol{C}'\boldsymbol{P}^*_{21,t+1}\boldsymbol{\Phi} + \boldsymbol{\Phi}'\boldsymbol{P}^*_{21,t+1}\boldsymbol{\Phi} \\ &- \left(\boldsymbol{C}'\boldsymbol{P}^*_{11,t+1} + \boldsymbol{\Phi}'\boldsymbol{P}^*_{21,t+1}\right) \\ &\times \boldsymbol{B}\boldsymbol{\Delta}^{-1}_{11,t+1}\boldsymbol{B}' \left(\boldsymbol{P}^*_{11,t+1}\boldsymbol{C} + \boldsymbol{P}^*_{12,t+1}\boldsymbol{\Phi}\right).\end{aligned} \tag{31}$$

where

$$\begin{aligned}\boldsymbol{P}^*_{11,t+1} &= \left(\boldsymbol{I}_n + \boldsymbol{P}_{11,t+1}\boldsymbol{C}\boldsymbol{\Theta}^{-1}_{t+1}\boldsymbol{C}'\right)\boldsymbol{P}_{11,t+1} \\ \boldsymbol{P}^*_{12,t+1} &= \left(\boldsymbol{I}_n + \boldsymbol{P}_{11,t+1}\boldsymbol{C}\boldsymbol{\Theta}^{-1}_{t+1}\boldsymbol{C}'\right)\boldsymbol{P}_{12,t+1} \\ \boldsymbol{P}^*_{22,t+1} &= \boldsymbol{P}_{22,t+1} + \boldsymbol{P}_{21,t+1}\boldsymbol{C}\boldsymbol{\Theta}^{-1}_{t+1}\boldsymbol{C}'\boldsymbol{P}_{12,t+1}\end{aligned}$$

with $\left(\boldsymbol{P}^*_{12,t+1}\right)' = \boldsymbol{P}^*_{21,t+1}$ when $(\boldsymbol{P}_{12,t+1})' = \boldsymbol{P}_{21,t+1}$.

4 Some numerical results

In this section some numerical results are presented. The classical MacRae problem with one state, one control and two periods, extensively used in the control literature, see e.g.[23] and [24], has been recently 'robustized' in [36] and [37]. It represents a wonderful workhorse when dealing with new problems. it is suitable for hand calculations and it is wonderful to debug computer codes, see, e.g., [22] and [35] . The robust version of this problem may be restated as: extremize, i.e. maximize with respect to u_t and minimize with respect to

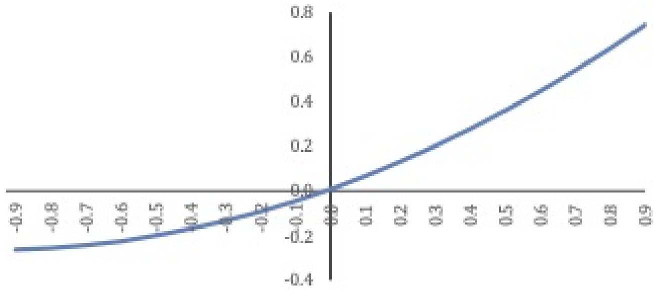

Fig. 1. Robust control at time zero for the MacRae problem determined assuming various ϕ's.

ω_{t+1} the objective function

$$J = E_0 \left[\sum_{t=1}^{2} \left(q_t y_t^2 + w_{t-1} y_{t-1} u_{t-1} + r_{t-1} u_{t-1}^2 \right) - \theta \sum_{t=1}^{2} \beta^t \omega_t^2 \right] \quad \text{with} \quad 0 < \beta < 1 \tag{32}$$

subject to

$$y_{t+1} = a y_t + b u_t + c(\varepsilon_{t+1} + \omega_{t+1}) \quad \text{for} \quad t = \ 0,\ 1. \tag{33}$$

with all the symbols defined as before, except for the fact that they are now scalar quantities instead of generic arrays, and the planning horizon limited to only two periods. The colored noise ε_{t+1} is assumed to follow a first order autoregressive process, i.e. $\varepsilon_{t+1} = \phi \varepsilon_t + \xi_{t+1}$ with $\xi_{t+1} iid\ \mathcal{N}(0,1)$. The system parameters are assumed perfectly known and in addition $\varepsilon_t = 0.3$, $q_t = \beta^t q_0$, $r_t = \beta^t r_0$, $w_t = \beta^t w_0$, $\beta = .99$ and $\theta = 10000$.

When the parameters are [22]

$$a = .7, b = -.5, c = 3.5, q_0 = r_0 = 1 \text{ and } w_0 = 1 \tag{34}$$

the control selected by the controller is $u_0 = .007$, compared to $u_0 = .006$ for the traditional quadratic-linear approach, when $\phi = 0.0$. This corresponds to the usual robust control formula used in the macroeconomic literature. Namely, it is equivalent to the assumption that ε_{t+1} is white noise. When the system error is colored with transition parameter $\phi = 0.1$ the appropriate robust control increases to $u_0 = .066$ and it skyrockets to $u_0 = .744$, around 100 times the usual robust control value when $\phi = 0.9$ (see Fig. 1). In this model the assumed positive shock ($\varepsilon_t = 0.3$) represents 'bad news' for the controller. When the state is inflation rate and the control is the interest rate charged by the Central Bank, a positive transition parameter means that yesterday positive random shock is going to adversely affect future inflation and

[22] This is the parameter set originally used in [23].

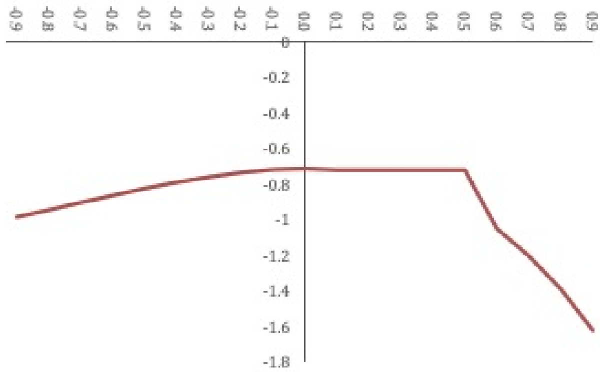

Fig. 2. Expected cost associated for the MacRae problem when robust controls at time zero are determined assuming various ϕ's.

the controller needs to adopt a stronger policy to partially offset these consequences. How much stronger this reaction will be it depends upon the size of the transition parameter. Even a small transition parameter, in the present case $\phi = 0.1$, may result in a meaningful increase of the control. In this situation a negative transition parameter decreases the need of a stronger control. In this case the higher inflation today is associated to lower expected inflation in the future. Then robust control turns negative. It is interesting to notice, however, that the controller reacts less strongly to 'good forecast', i.e. knowing that yesterdays shock is to some extent offset today, than to 'bad forecasts', i.e. knowing that yesterdays shock will keep generating negative affects today. This is due to the fact that 'good forecast for today are short lived in the sense that they are associated with 'bad forecast for tomorrow as apparent from the correlogram of the process governing the system noise. On the other hand 'bad forecasts' for today are 'bad forecasts' for ever.

The expected cost associated with the robust control selected when the system error is white noise, i.e. $\phi = 0.0$, is $-.7128$. It decreases to $-.7197$ for $\phi = 0.1$ and remains in that region for values of the transition parameter below 0.6. This means that by applying the correctly determined robust control in the presence of colored system noise the controller is able to effectively minimize the consequences of 'bad news' on social welfare in many cases. The expected cost decreases sharply for high values of the transition parameter (Fig. 2).

When the well known permanent income model is used, with the parameter estimates in [18] and the preference parameter governing the curvature of the utility function $\mu_b = 0$,[23] the robust control selected by the controller is $u_0 = -44.9195$, when $\theta = 10000$ and the system error is white noise, compared to $u_0 = -44.9205$ for the traditional quadratic-linear approach.The difference between the two controls is what [12, p. 47] refer to as precautionary saving induced by the consumer preference for robustness. In this model a positive

[23] See, e.g., [9], [10], [12], [17], [18] , [36] or [37] for a description of the model.

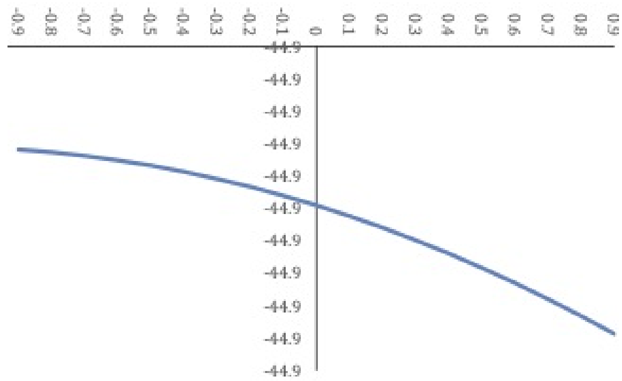

Fig. 3. Control at time zero for the permanent income model determined in the presence of various ϕ's.

shock of $\varepsilon_t = 0.3$ positively affects the consumer stochastic endowment process. When the system error is colored with transition parameter $\phi = 0.1$ the appropriate control decreases to $u_0 = -44.9212$ and it reaches $u_0 = -44.9393$ when $\phi = 0.9$ (see Fig. 3). By knowing that yesterday positive random shock is going to affect positively expected future endowments the consumer is willing to decrease her/his savings. In this, more complicated, problem the changes in the controls are not as striking as in the simple MacRae problem. One reason is that the parameters in the volatility matrix $\boldsymbol{C}$ used in [18] are .11-.15, much smaller than the value used for the MacRae problem. In the presence of a negative transition parameter, the consumer reacts less strongly by increasing the amount of precautionary savings up to $u_0 = -44.9195$ when $\phi = -0.9$. As in the simpler model, the reaction associated to a negative transition parameter is less vigorous than that associated with a positive transition parameter.

The expected cost associated with the robust control selected when the system error is white noise is 13919.38. It remains in the interval 13919.37-13919.38 for all values of the transition parameter. This means that by applying the correctly determined robust control in the presence of colored system noise the consumer is able to effectively minimize the consequences of a colored noise on her/his welfare (Fig. 4).

5 Conclusion

In this paper a standard robust control problem in discrete time with unstructured uncertainty à la Hansen and Sargent, i.e. a nonparametric set of additive mean-distorting model perturbations, where the decision maker is assumed to be "probabilistically sophisticated" is considered. The case of a colored model error (or system disturbance) term is treated. An augmented state vector including both the original states and the colored error term is defined and the Bellman equation for the augmented system is introduced.

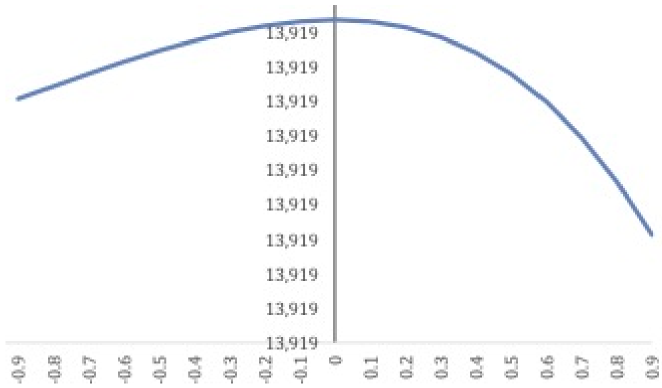

Fig. 4. Expected cost associated for the permanent income model when robust controls at time zero are determined assuming various ϕ's.

Then the new θ-constrained worst-case controls for the decision maker and malevolent nature are derived. It turns out that both of them can be easily computed. The decision maker 'colored' control is equal to the 'white noise' case plus a corrections factor depending upon the transition parameter in the noise equation and $\boldsymbol{P}_{12}$. Similarly, the 'colored' malevolent nature control is equal to the 'white noise' case, when it is taken into account the new decision maker's decision, plus a corrections factor depending upon Φ and $\boldsymbol{P}_{12}$. When the transition parameter is positive, a positive shock in the current period negatively affects future forecasts and the controller in the 'robustized' version of the MacRae problem, described in [36] and [37], must apply a much stronger control. The same situation denotes higher expected future endowments for the consumer in the permanent income model. Then the consumer is willing to decrease her/his savings. A negative transition parameter induces the controller in both models to react in the opposite direction. However this reaction is not as strong as in the presence of a positive transition parameter.

References

[1] B.D.O. Anderson and J.U. Moore. *Optimal filtering.* Prentice Hall, Englewood Cliffs, NJ, 1979.

[2] Tamer Basar and Pierre Bernhard. *H^∞-optimal control and related minimax design problems: A dynamic game approach.* Birkhäuser, MA-Basel-Berlin, 1st edition, 1991.

[3] A. Ben-Tal, L. El Ghaoui, and A. Nemirovsky. *Robust optimization.* Princeton University Press, Princeton, NJ, 2009.

[4] Pierre Bernhard. Survey of linear quadratic robust control. *Macroeconomic Dynamics*, 6(1):19–39, 2002.

[5] D. Bertsimas, D.B. Brown, and C. Caramanis. Theory and applications of robust optimization. *SIAM Review*, 53(3):464–501, 2011.

[6] E. de Klerk, D. Kuhn, and K. Postek. Distributionally robust optimization with polynomial densities: Theory, models and algorithms. *Mathematical Programming, Series B*, 181(265-296), 2020.

[7] Marc P. Giannoni. Does model uncertainty justify caution? robust optimal monetary policy in a forward-looking model. *Macroeconomic Dynamics*, 6(1):111–144, 2002.

[8] Marc P. Giannoni. Robust optimal policy in a forward-looking model with parameter and shock uncertainty. *Journal of Applied Econometrics*, 22(1):179–213, 2007.

[9] Lars P. Hansen and Thomas J. Sargent. Acknowledging misspecification in macroeconomic theory. *Review of Economic Dynamics*, 4(3):519–535, 2001.

[10] Lars P. Hansen and Thomas J. Sargent. Robust control of forward-looking models. *Journal of Monetary Economics*, 50(3):581–604, 2003.

[11] Lars P. Hansen and Thomas J. Sargent. Recursive robust estimation and control without commitment. *Journal of Economic Theory*, 136(1):1–27, 2007.

[12] Lars P. Hansen and Thomas J. Sargent. *Robustness.* Princeton University Press, Princeton, NJ-Oxford, 2007.

[13] Lars P. Hansen and Thomas J. Sargent. Wanting robustness in macroeconomics. In Benjamin M. Friedman and Michael Woodford, editors, *Handbook of Monetary Economics*, volume 3B, pages 1097–1157. Elsevier, San Diego, CA, The Netherlands, 2010.

[14] Lars P. Hansen and Thomas J. Sargent. Robustness and ambiguity in continuous time. *Journal of Economic Theory*, 146:1195–1223, 2011.

[15] Lars P. Hansen and Thomas J. Sargent. Sets of models and prices of uncertainty. *NBER*, 22000, National Bureau of Economic Research, Inc.. 2016.

[16] Lars P. Hansen and Thomas J. Sargent. Macroeconomic uncertainty prices when beliefs are tenuous. *Journal of Econometrics*, 223:222–250, 2021.

[17] Lars P. Hansen, Thomas J. Sargent, and Thomas D. Tallerini. Robust permanent income and pricing. *Review of Economic Studies*, 66(4):873–907, 1999.

[18] Lars P. Hansen, Thomas J. Sargent, and Neng E. Wang. Robust permanent income and pricing with filtering. *Macroeconomic Dynamics*, 6(1):40–84, 2002.

[19] Lars P. Hansen, Bálint Szőke, Lloyd S. Han, and Thomas J. Sargent. Twisted probabilities, uncertainty, and prices. *Journal of Econometrics*, 216:151–174, 2020.

[20] Andrew C. Harvey. *Time series models.* Philip Allan, Oxford, 1981.

[21] M.D. Intriligator. *Mathematical Optimization and Economic Theory.* Prentice Hall, Englewood Cliffs, NJ, 1971.

[22] David A. Kendrick. *Stochastic Control for Economic Models.* McGraw-Hill, New York, NY., 1981.

[23] Elizabeth C. MacRae. Linear decision with experimentation. *Annals of Economic and Social Measurement*, 1:437–447, 1972.

[24] Elizabeth C. MacRae. An adaptive learning role for multi-period decision problems. *Econometrica*, 43:893–906, 1975.

[25] Alexei Onatski and James H. Stock. Robust monetary policy under model uncertainty in a small model of the u.s. economy. *Macroeconomic Dynamics*, 6(1):85–110, 2002.

[26] B. P. G. Van Parys, D. Kuhn, P. J. Goulart, and M. Morari. Distributionally robust control of constrained stochastic systems. *Ieee T Automat Contr*, 61:430–442, 2016.

[27] Ian R. Petersen and Roberto Tempo. Robust control of uncertain systems: Classical results and recent developments. *Automatica*, 50(1315-1335), 2014.

[28] Hamed Rahimian and Sanjay Mehrotra. Frameworks and results in distributionally robust optimization. *Open Journal of Mathematical Optimization*, 3(4):1–85, 2022.

[29] Berç Rustem. A constrained min-max algorithm for rival models of the same economic system. *Mathematical Programming*, 53(1-3):279–295, 1992.

[30] Berç Rustem. Robust min-max decisions for rival models. In David A. Belsley, editor, *Computational technique for econometrics and economic analysis.* Kluwer Academic Publishers, Dordrecht, NL, 1994.

[31] Berç Rustem. *Algorithms for nonlinear programming and multiple objective decisions.* Wiley, New York, NY, 1998.

[32] Berç Rustem and Melendres A. Howe. *Algorithms for worst-case design with applications to risk management.* Princeton University Press, Princeton, NJ, 2002.

[33] Robert Tetlow and Peter von zur Muehlen. Robust monetary policy with misspecified models: does model uncertainty always call for attenuated policy? *Journal of Economic Dynamics and Control,*, 25(6-7):911–949, 2001.

[34] Robert Tetlow and Peter von zur Muehlen. Simplicity versus optimality: the choice of monetary policy rules when agents must learn. *Journal of Economic Dynamics and Control*, 25(1-2):245–279, 2001.

[35] Marco P. Tucci. *The Rational Expectation Hypothesis, Time-Varying Parameters and Adaptive Control: A Promising Combination?* Springer, Dordrecht. The Netherlands, 2004.

[36] Marco P. Tucci. Understanding the difference between robust control and optimal control in a linear discrete-time system with time-varying parameters. *Computational Economics*, 27(4):533–558, 2006.

[37] Marco P. Tucci. How robust is robust control in discrete time? *Computational Economics*, 58(4):279–309, 2021.

[38] Marco P. Tucci. A critical introduction to the usual robust control framework in macroeconomics. *Computational Economics*, doi.org/10.1007/s10614-023-10454-8, 2023.

[39] von zur P. Muehlen. Activist versus non-activist monetary policy: Optimal policy rules under extreme uncertainty. reprinted in Finance and Economics Discussion Series No 2002-02 (2001), Federal Reserve USA, downloadable from www.federalreserve.gov/pubs/feds/, 1982.