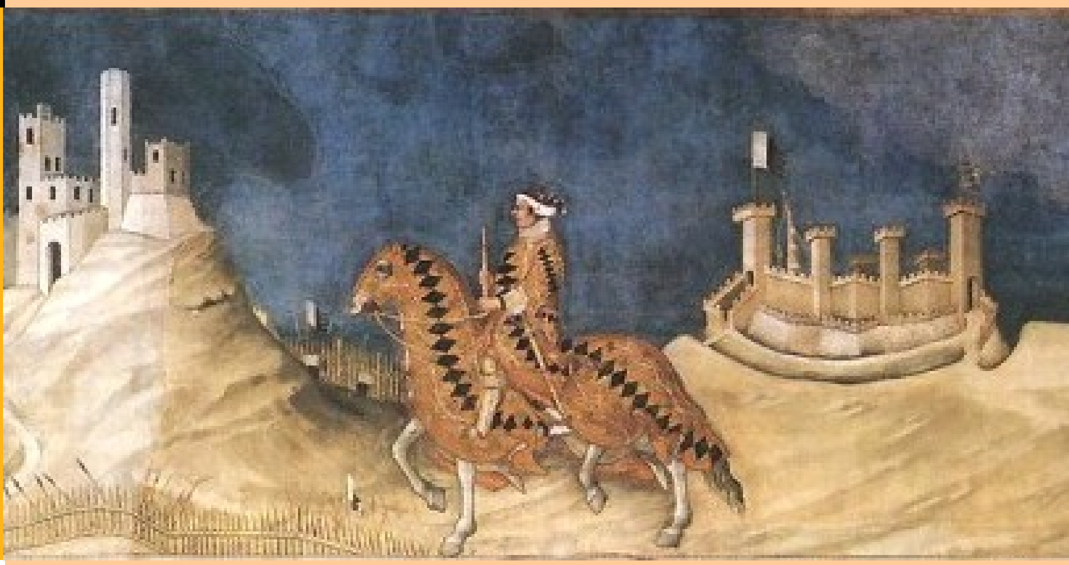
UNIVERSITÀ DI SIENA
1240

QUADERNI DEL DIPARTIMENTO
DI ECONOMIA POLITICA E STATISTICA

Stefano Vannucci

Network geometry and the scope of the median voter theorem

n. 704 – Gennaio 2015



Network geometry and the scope of the median voter theorem

Stefano Vannucci
Department of Economics and Statistics,
University of Siena
Piazza S.Francesco 7, 53100 Siena, Italy

December 9, 2014

Abstract

It is shown that the median voter theorem for committee-decisions holds over a full unimodal preference domain whenever

(i) the underlying median interval space satisfies interval anti-exchange and

(ii) unimodality is defined with respect to the incidence-geometry of the relevant outcome space or network.

Thus, in particular, the interval spaces canonically induced by trees do support the median voter theorem on their own full unimodal preference domains. Conversely, validity of the median voter theorem on the full unimodal preference domain of a certain median interval space on a discrete outcome space requires that the graph canonically induced by that interval space be precisely a tree.

JEL Classification : D71

1 Introduction

The so-called 'median voter theorem' says that if the voters' preferences are 'unimodal' then median outcomes are also 'majority winners': it refers in fact to an entire family of results concerning both *voting in committees* and *elections of candidates in modern representative democracies*, and relying on several distinct specifications of the domain of 'unimodal' preferences and of the notion of 'majority winner'. Accordingly, one may sensibly distinguish two varieties of median voter results:

(i) **'Median voter theorem(s)' for committees**: if voters' preferences are '**unimodal**' i.e. have a unique maximum and '**respect compromises**' then **the median** of the alternative outcomes actually chosen by single voters is a Condorcet winner i.e. is preferred to any other outcome by some majority of voters.

(ii) **'Median voter theorem(s)' for elections of a candidate in a representative democracy:** if candidates choose their platforms in the outcome space just in order to win the election and are able to predict the distribution of voters' choices, and voters' preferences are '**unimodal**' i.e. have a unique maximum and '**respect compromises'**, then the only (strict) Nash equilibrium of the strategic platform-selection game played by candidates is the profile of platforms where *each* candidate selects **the median** of the alternative outcomes actually chosen by single voters.

The paradigm of (i) is the classic result by Black (1948) (but see also, Moulin (1980), Wendell, McKelvey (1981), Hansen, Thisse (1981), Moulin (1983), Bandelt, Barthélemy (1984), Bandelt (1985), Danilov (1994)). The paradigm of (ii) is Downs (1957) (who explicitly draws on ideas advanced by Hotelling (1929) in his classic analysis of spatial competition in duopoly; but see also Wendell, McKelvey (1981), and especially Roemer (2001) for an extensive, thorough discussion of several variants of 'median voter theorems' of that type).

It should be emphasized at the outset that the reason why choosing the median outcome among voters' choices as a platform is the common, unique (strict) Nash equilibrium strategy for each candidate *is due precisely to the fact that under the given assumptions that platform/outcome is a Condorcet winner*.

Thus, the formulations of 'median voter theorems' presented above should make clear two main points, namely:

1. The 'representative democracy'- variety relies on the 'committee'-

variety of median voter theorem(s) (plus some supplementary more or less disputable hypotheses on the number, motivation and/or forecasting ability of candidates and their possibly supporting parties: see Roemer (2001) on those issues and other related matters).

2. The ‘committee’- variety of median voter theorems relies heavily on an underlying geometric structure of the outcome space that makes it possible to sensibly define a ***median*** *(and in particular a* ***unique*** *median for any odd outcome-sample),* to give a sound meaning to the notion of a **compromise** between any two outcomes, thereby making it also possible to define **unimodal** preferences.

It follows that the **network geometry** of the outcome space is a key issue when discussing the scope of median voter theorems. The classic versions of the ‘committee’-variety assume outcome spaces consisting of a bounded line or chain (Black (1948), Moulin (1980)), a finite tree under metric unimodal preferences (Wendell, McKelvey (1981)), an arbitrary median algebra or median semilattices under metric preference profiles (Bandelt, Barthélemy (1984), or a bounded tree under preference profiles of *linear orders* (Danilov (1994)).

The present paper is devoted to a wider analysis of the scope of the ‘committee’-variety of median voter theorems by focusing on **unimodal domains of incidence-type -as opposed to metric- type-** in an arbitrary **median interval space** (see van de Vel (1993) and Coppel (1998) for an extensive in-depth treatment of interval spaces).

2 On the scope of the ‘median voter theorem’: basic definitions and preliminaries

2.1 The structure of the outcome space: defining median interval spaces and related structures

Let $\mathcal{I} = (X, I)$ be the **interval space** of alternative outcomes, i.e. X is an arbitrary nonempty set and I an *interval function on* X, namely $I : X^2 \to \mathcal{P}(X)$ is a function that satisfies the following conditions:

I-(i) (**Extension**): $\{x, y\} \subseteq I(x, y)$ for all $x, y \in X$,

I-(ii) (**Symmetry**): $I(x, y) = I(y, x)$ for all $x, y \in X$.

In particular, will be mostly concerned with *idempotent* interval spaces i.e. with interval spaces that also satisfy

(**Idempotence**): $I(x,x)=\{x\}$ for all $x\in X$.

A subset $Y\subseteq X$ is *$\mathcal{I}$-convex* iff $I(x,y)\subseteq Y$ for all $x,y\in Y$. For any $Y\subseteq X$, the *$\mathcal{I}$-convex hull* of Y - denoted $co_{\mathcal{I}}(Y)$- is the smallest $\mathcal{I}$-convex superset of Y, namely $co_{\mathcal{I}}(Y)=\bigcap\{A\subseteq X : A$ is *$\mathcal{I}$-convex* and $A\supseteq Y\}$.

An interval space $\mathcal{I}=(X,I)$ is *convex* (or *interval-monotonic*) if I also satisfies

(**Convexity**): $I(x,y)$ is $\mathcal{I}$-convex for all $x,y\in X$.

Observe that Idempotence and Convexity are indeed mutually independent properties of interval spaces. To confirm that statement, consider interval spaces $\mathcal{I}_1=(X,I_1)$, $\mathcal{I}_2=(\{x,y,v,z\},I_2)$ where $\#X>1$, $\#\{x,y,v,z\}=4$, $I_1(a,b)=X$ for all $a,b\in X$, while $I_2(x,y)=\{x,y,z\}$, $I_2(y,z)=\{y,v,z\}$, and $I_2(a,b)=\{a,b\}$ for all $a,b\in X$ such that $\{x,y\}\neq\{a,b\}\neq\{y,z\}$. It is immediately checked that $\mathcal{I}_1$ is convex but not idempotent, while $\mathcal{I}_2$ is idempotent but not convex since $\{y,z\}\subseteq I_2(x,y)$ and $v\in I_2(y,z)\setminus I_2(x,y)$.

Furthermore, an *idempotent* interval space $\mathcal{I}=(X,I)$ is said to be a **median space** if I satisfies the following

(**Median property**): for all $x,y,z\in X$, $|(I(x,y)\cap I(y,z)\cap I(x,z))|=1$

The common point of the three intervals defined by each pair of any three points x,y,z in a median interval space $\mathcal{I}=(X,I)$ is said to be the *median* of those points, that therefore defines a ternary operation -the median μ_I- on X: the pair $\mathcal{M}_I=(X,\mu_I)$ is the **ternary algebra induced by median interval space $\mathcal{I}$.**

It is well-known that any median interval space is also idempotent and convex (see Mulder (1980), Theorem 3.1.4).

An interval space $\mathcal{I}=(X,I)$ is **discrete** if $I(x,y)$ is finite for all $x,y\in X$.

The following property will play a key role in the ensuing analysis

(**Interval Anti-Exchange (IAE)**): for all $x,y,v,z\in X$ such that $x\neq y$, if $x\in I(y,v)$ and $y\in I(x,z)$ then $x\in I(v,z)$.

It should be noticed here that Interval Anti-Exchange, Idempotence and Convexity are mutually independent properties of an interval space[1].

[1] To check that statement, consider the following interval spaces: (i)$(X=\{x,y,u,v\},I)$ with $I(x,y)=\{x,u,y\}$, $I(u,y)=\{u,v,y\}$ and $I(a,b)=\{a,b\}$ for all $\{a,b\}\notin\{\{x,y\},\{u,y\}\}$, which is by construction idempotent and can be easily shown to satisfy IAE, but is clearly not convex; (ii) $(X=\{x,y\},I)$ with $I(x,x)=I(x,y)=\{x,y\}$, $I(y,y)=\{y\}$: that interval space is not idempotent but -as it is easily seen- it satisfies IAE and is obviously convex; (iii)$(X=\{x,y,z\},I)$ with $I(x,z)=I(y,z)=\{x,y,z\}$, and

A few supplementary basic notions are also to be introduced here.

A **median algebra** is a pair $\mathcal{M} = (X, \mu)$ where X is a set and $\mu : X^3 \to X$ is a ternary operation on X -the *median operation*- that satisfies the following three properties:

MA(i) $\mu(x,x,y) = x$ for all $x, y \in X$, MA(ii) $\mu(x,y,z) = \mu(y,x,z) = \mu(y,z,x)$ for all $x, y, z \in X$, MA(iii) $\mu(\mu(x,y,z),u,v) = \mu(x,\mu(y,u,v),\mu(z,u,v))$ for all $x, y, z, u, v \in X$.

The **interval space** $\mathcal{I}_\mu = (X, I_\mu)$ **induced by median algebra** $\mathcal{M} = (X, \mu)$ is defined as follows: for each $x, y \in X$,

$I_\mu(x,y) = \{z \in X : \mu(x,z,y) = z\}$.

A median algebra is **discrete** iff $I_\mu(x,y)$ is finite for all $x, y \in X$.

Let us now consider an ordered pair $\mathcal{X} = (X, \leqslant)$ where is $\leqslant$ a reflexive, transitive and antisymmetric binary relation on X, and denote by $\vee$ and $\wedge$ the *least-upper-bound* and *greatest-lower-bound* binary *partial* operations on X as induced by $\leqslant$, respectively. Moreover, for any $x, y \in X$, x is said to *cover* y -written $y \ll x$- if $y \leqslant x$ and there is no $z \in X \smallsetminus \{x, y\}$ such that $y \leqslant z \leqslant x$.

The ordered pair $\mathcal{X} = (X, \leqslant)$ is a **median semilattice** if and only if

(i) $x \wedge y$ is well-defined in X for all $x, y \in X$ i.e. $\mathcal{X}$ is a *meet-semilattice;*

(ii) for all $u \in X$, and for all $x, y, z \in X$ such that u is an upper bound of $\{x, y, z\}$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (or, equivalently, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$) holds i.e. $(\downarrow u, \leqslant_{|\downarrow u})$ -where $\leqslant_{|\downarrow u}$denotes the restriction of $\leqslant$ to $\downarrow u = \{x \in X : x \leqslant u\}$- is a *distributive lattice*[2] i.e. $\mathcal{X}$ itself is a *lower distributive meet-semilattice*;

(iii) for all $x, y, z \in X$ if $x \vee y$, $y \vee z$ and $x \vee z$ exist, then $(x \vee y) \vee z$ also exists i.e. $\mathcal{X}$ satisfies the *coronation (or join-Helly) property.*

It is easily checked that if $\mathcal{X} = (X, \leqslant)$ is a median meet-semilattice then the partial function $\mu : X^3 \to X$ defined as follows: for all $x, y, z \in X$

$$\mu(x,y,z) = (x \wedge y) \vee (y \wedge z) \vee (x \wedge z)$$

is indeed a *well-defined ternary operation* X*,* the *median* of $\mathcal{X}$.

A partially ordered set $\mathcal{X} = (X, \leqslant)$ is **discrete** if it has no infinite bounded chain (i.e. there are no $Y \subseteq X$ and $a, b \in Y$ such that $x \leqslant y$ or $y \leqslant x$ for all $x, y \in Y$, $a \leqslant x \leqslant b$ for all $x \in Y$, and Y is an infinite set).

$I(a,b) = \{a,b\}$ for all $\{a,b\} \notin \{\{x,z\},\{y,z\}\}$, which is by construction idempotent and convex but fails to satisfy IAE since $x \in I(y,z)$, $y \in I(x,z)$ and $x \notin I(z,z)$.

[2] A poset $(Y, \leqslant)$ is a *distributive lattice* iff, for any $x, y, z \in X$, $x \wedge y$ and $x \vee y$ exist, and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (or, equivalently, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$).

It is worth recalling here that the notion of median semilattice -under the alternative label of 'ternary distributive semi–lattice'- is due to Avann (1948), while the notion of median algebra is mainly due to Sholander (1952): both of these developments did rely on earlier, seminal ideas introduced by Birkhoff, Kiss (1947) (see Bandelt, Hedlíková (1983) for more details).

A **(simple) graph** is a pair $G = (X, E)$ where X is a set -the set of vertices- and $E \subseteq \{\{x, y\} : x, y \in X\}$ denotes the set of edges: the *order* of G is $|X|$ (where $|X|$ denotes the cardinality of set X), and for each vertex $x \in X$ the *degree* of x is $\deg(x) = |\{y \in X : \{x, y\} \in E\}|$. A graph is *regular* if $\deg(x) = \deg(y)$ for all $x, y \in X$. For any $x, y \in X$, a (simple, elementary) *path* joining x and y in G is a bounded well-ordered set of pairwise distinct edges $\{E_i\}_{i\in I}$ of G such that $x \in E_0$, $y \in E_{i^*}$ (where E_0 and E_{i^*} denote the minimum and maximum edges of path $\{E_i\}_{i\in I}$), $|E_i \cap E_{\min\{j\in I:i<j\}}| = 1$, and for each $z \in \cup_{i\in I} E_i$, $z \in E_i \cap E_j$ only if $E_i = E_j$ or $E_j = E_{\min\{j\in I:i<j\}}$(i.e. no repetition of vertices allowed in *distinct nonconsecutive* edges). The *length* of path $\{E_i\}_{i\in I}$ in G is given by $|I|$. A *cycle* of graph G is a path of length $l > 1$ joining x and $y \in X$ in G with $x = y$. A graph is said to be *acyclic* (or a *forest*) if it has no cycles, and *connected* if for each $x, y \in X$ there exists a path joining x and y in G. A **tree** is an acyclic connected graph: clearly, for any two vertices of a tree there exists a unique path joining them. For any $x, y \in X$ a *geodesic* of G for that pair is a path of minimum length joining x and y in G: if G is a connected graph the length of a geodesic between any two vertices x, y defines a *distance* between them, denoted $d_G(x, y)$. Notice that, for any $x, y \in X$, a vertex $z \in X$ belongs to *some* geodesic joining x and y if and only if $d_G(x, y) = d_G(x, z) + d_G(z, y)$. If in particular G is a tree then there exists precisely *one* path joining any two vertices, and that unique path is also *the* geodesic of the tree for them. An **(extended) median** of a finite set $\{x_1, ..., x_k\} \subseteq X$ of vertices of G is a vertex $m_G \in X$ such that $m_G \in \arg\min_{y\in X} \sum_{i=1}^{k} d_G(y, x_i)$.

The **graph** $G_I = (X, E_I)$ **induced by interval space** $\mathcal{I} = (X, I)$ is defined as follows: for each $x, y \in X$, $\{x, y\} \in E_I$ if and only if $I(x, y) = \{x, y\}$.

The **graph** $G_\mu = (X, E_\mu)$ **induced by median algebra** $\mathcal{M} = (X, \mu)$ is defined as follows: for each $x, y \in X$, $\{x, y\} \in E_\mu$ if and only if $I_\mu(x, y) = \{x, y\}$: if $G_\mathcal{M} = (X, E_\mu)$ is connected, then the median algebra $\mathcal{M} = (X, \mu)$ is said to be *discrete*.

The **covering graph** $G_{\leqslant} = (X, E_{\leqslant})$ of a discrete partially ordered set $\mathcal{X} = (X, \leqslant)$ is defined by the following rule: for each $x, y \in X$, $\{x, y\} \in E_{\leqslant}$ if and only if either $x \ll y$ or $y \ll x$.

The **interval space** $= (X, I_G)$ **induced by graph** $G = (X, E)$ is defined as follows: for each $x, y \in X$,

$I_G(x, y) = \{z \in X : d_G(x, y) = d_G(x, z) + d_G(x, z)\}$,

namely a vertex belongs to the interval between x and y if and only if it lies on *some* geodesic joining x and y.

It is well-known that the interval spaces induced by trees are median (see Lemma 7 below for a sketch of the proof).

A **median graph** is a graph G whose induced interval space $\mathcal{I}_G$ is a *median* interval space as defined above.

2.2 Unimodal preferences on interval spaces: unimodality as a incidence-based notion

Let $\succcurlyeq$ denote a total preorder i.e. a reflexive, connected and transitive binary relation on X (we shall denote by $\succ$ and $\sim$ its asymmetric and symmetric components, respectively). Then, $\succcurlyeq$ is said to be unimodal with respect to interval space $\mathcal{I} = (X, I)$ - or $\mathcal{I}$-**unimodal** - if and only if

U-(i) there exists a *unique maximum* of $\succcurlyeq$ in X, its *top* outcome -denoted *top*$(\succcurlyeq)$- and

U-(ii) for all $x, y, z \in X$, if $z \in I(x, y)$ then $\{u \in X : z \succcurlyeq u\} \cap \{x, y\} \neq \varnothing$.

We denote by $U_{\mathcal{I}}$ the set of all $\mathcal{I}$-*unimodal* total preorders on X. An N-profile of $\mathcal{I}$-unimodal total preorders is a mapping from N into $U_{\mathcal{I}}$. We denote by $U_{\mathcal{I}}^N$ the set of all N-profiles of $\mathcal{I}$-unimodal total preorders. Notice that $\mathcal{I}$-unimodality is an *incidence-theoretic* notion which should be contrasted with *metric* unimodality that relies on a metric space (X, d^*), requiring again a unique maximum x^* but positing $y \succcurlyeq_x^* z$ if and only if $d^*(x, y) \leqslant d^*(x, z)$.

2.3 Voting rules and their properties

Let $N = \{1, .., n\}$ denote the finite population of voters (we assume $|N| \geq 2$ to avoid tedious qualifications). A **voting rule** for (N, X) is a function $f : X^N \to X$. A voting rule f is (simply) **strategy-proof** *on* $U_{\mathcal{I}}^N$ iff for all $\mathcal{I}$-unimodal N-profiles $(\succcurlyeq_i)_{i \in N} \in U_{\mathcal{I}}^N$, and for all $i \in N$, $y_i \in X$, and $(x_j)_{j\in N} \in X^N$ such that $x_j = top(\succcurlyeq_j)$ for each $j \in N$, $f((x_j)_{j\in N}) \succcurlyeq_i f((y_i,(x_j)_{j\in N\smallsetminus\{i\}}))$. Moreover, a voting rule f is **coalitionally strategy-proof** *on* $U_{\mathcal{I}}^N$ iff for all $\mathcal{I}$-unimodal N-profiles $(\succcurlyeq_i)_{i\in N} \in U_{\mathcal{I}}^N$, and for all $C \subseteq N$, $(y_i)_{i\in C} \in X^C$, and $(x_j)_{j\in N} \in X^N$ such that $x_j = top(\succcurlyeq_j)$ for each $j \in N$, there exists $i \in C$ with $f((x_j)_{j\in N}) \succcurlyeq_i f((y_i)_{i\in C},(x_j)_{j\in N\smallsetminus C}))$. Finally, a voting rule $f : X^N \to X$ is $\mathcal{I}$-**monotonic** iff for all $i \in N$, $y_i \in X$, and $(x_j)_{j\in N} \in X^N$, $f((x_j)_{j\in N}) \in I(x_i, f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}))$. An outcome $x \in X$ is a **Condorcet winner (CW)** at preference profile $(\succcurlyeq_i)_{i\in N} \in U_{\mathcal{I}}^N$ iff for all $y \in X$, $n_{xy}((\succcurlyeq_i)_{i\in N}) \geq n_{yx}((\succcurlyeq_i)_{i\in N})$ where, for any $u, v \in X$, $n_{uv} = \{i \in N : u \succcurlyeq_i v\}$. The set of all Condorcet winners at profile $(\succcurlyeq_i)_{i\in N} \in U_{\mathcal{I}}^N$ is also written $CW((\succcurlyeq_i)_{i\in N})$. A voting rule $f : X^N \to X$ is a **CW-selection** on $U_{\mathcal{I}}^N$ iff -for all N-profiles $(\succcurlyeq_i)_{i\in N} \in U_{\mathcal{I}}^N$- $f((top(\succcurlyeq_i)_{i\in N})) \in CW((\succcurlyeq_i)_{i\in N})$

2.4 Some basic results about median interval spaces and related structures

We collect in that subsection some well-known basic facts to be used below about the tight connection between median interval spaces, median algebras, median semilattices and median graphs as established in some early, fundamental works by Sholander (1952, 1954(a), 1954(b)), and Avann (1961).

Proposition 1 *(Sholander (1954 (a))) If $\mathcal{M} = (X, \mu)$ is a median algebra then its induced interval space $\mathcal{I}_\mu = (X, I_\mu)$ is median. Conversely, if $\mathcal{I} = (X, I)$ is a median interval space then the ternary operation $\mu_I : X^3 \to X$ as defined by the rule $\mu_I(x, y, z) = m$ such that $\{m\} = I(x, y) \cap I(y, z) \cap I(x, z)$ induces a median algebra $\mathcal{M}_I = (X, \mu_I)$;*

Proposition 2 *(Sholander (1954(b), and (1952))): If $\mathcal{X} = (X, \leqslant)$ is a median semilattice then the ternary operation $\mu_{\mathcal{X}}$ on X as defined by the rule $\mu_{\mathcal{X}}(x, y, z) = (x\wedge y)\vee(y\wedge z)\vee(x\wedge z)$ induces a median algebra $\mathcal{M}_{\mathcal{X}} = (X, \mu_{\mathcal{X}})$. Conversely if $\mathcal{M} = (X, \mu)$ is a median algebra then for any $x \in X$ positing $y \leqslant_{x,\mu} z$ if and only if $\mu(x, y, z) = y$ and defining $x \wedge' y = \inf(\leqslant_{x,\mu}))\{x, y\}$ and $x \vee' y = \sup(\leqslant_{x,\mu}))\{x, y\}$ (where $\vee'$ is a partial operation) induces a median semilattice $\mathcal{X}_{x,\mu} = (X, \leqslant_{x,\mu})$ such that for all $x, y, z \in X$, $(x \wedge' y) \vee (y \wedge' z) \vee (x \wedge' z) = \mu(x, y, z)$ (hence the median of the semilattice does not depend on the choice of x).*

Proposition 3 *(Avann (1961)): The covering graph* $G_{\mathcal{X}} = (X, E_{\leqslant})$ *of a discrete median semilattice* $\mathcal{X} = (X, \leqslant)$ *is a median graph, and the graph* $G_{\mathcal{M}} = (X, E_{\mu})$ *induced by a discrete median algebra* $\mathcal{M} = (X, \mu)$ *is also a median graph. Moreover,* $G_{\mathcal{X}} = G_{\mathcal{M}_{\mathcal{X}}}$.

Conversely, for every median graph $G = (X, E)$*, there exists a unique median algebra* $\mathcal{M} = (X, \mu)$ *hence a unique median interval space* $\mathcal{I} = (X, I)$ *such that* $G = G_{\mu} = G_I$ *(with* $\mu(x, y, z) = m$ *such that* $I_G(x, y) \cap I_G(y, z) \cap I_G(x, z) = \{m\}$ *for all* $x, y, z \in X$*).*

We shall also make use of the following result:

Proposition 4 *(Bandelt, Barthelémy (1984)) Let* $\mathcal{X} = (X, \leqslant)$ *be a discrete semilattice. Then* $\mathcal{X}$ *is a median semilattice if and only if for any positive integer* k *and any set* $\{x_1, ..., x_{2k+1}\} \subseteq X$ *, the (extended) median* $\mu^*(x_1, ..., x_{2k+1}) = \vee_{S \subseteq N, |S|=k+1} \wedge_{h \in S} x_h$ *is well-defined, and is the unique (extended) median of* $\{x_1, ..., x_{2k+1}\}$ *in the covering graph of* $\mathcal{X}$.

3 Median interval spaces and the scope of the 'median voter theorem'

We are now ready to state the main results of this paper concerning the equivalence of strategy-proofness and coalitional strategy-proofness of voting rules on the domain of all unimodal profiles. Our results rely on the following proposition that establishes the equivalence between *monotonicity* with respect to an arbitrary convex idempotent interval space $\mathcal{I}$ and *strategy-proofness on the corresponding (full) unimodal domain* $U_{\mathcal{I}}^N$.

Proposition 5 *(see Vannucci (2013)) Let* $\mathcal{I} = (X, I)$ *be a convex interval space. Then, a voting rule* $f : X^N \to X$ *is strategy-proof on the full unimodal domain* $U_{\mathcal{I}}^N$ *iff it is* $\mathcal{I}$*-monotonic.*

Theorem 6 *(Vannucci (2013)) Let $\mathcal{I} = (X, I)$ be a convex idempotent interval space that satisfies interval anti-exchange (IAE), and $f : X^N \to X$ a voting rule that is strategy-proof on the full unimodal domain $U_{\mathcal{I}}^N$. Then, f is also coalitionally strategy-proof on $U_{\mathcal{I}}^N$.*

Lemma 7 *Let $\mathcal{I} = (X, I)$ be a median interval space, $N = \{1, ..., 2k+1\}$ for some positive integer k, $x^* \in X$, and $\mu_{I,x^*} : X^N \longrightarrow X$ the extended n-ary median rule on X with respect to x^*. Then μ_{I,x^*} is $\mathcal{I}$-monotonic.*

Proof. To begin with, notice that the thesis reduces to showing that for each $x_N \in X^N$, $i \in N$, and $y \in X$, $\mu_{I,x^*}(x_N) = \mu_I(x_i, \mu_{x^*}(x_N), \mu_{x^*}(y, x_{N \smallsetminus \{i\}}))$, since by definition

$\mu_I(x_i, \mu_{I,x^*}(x_N), \mu_{I,x^*}(y, x_{N \smallsetminus \{i\}})) = m \in X$ such that $\{m\} = I(x_i, \mu_{I,x^*}(x_N)) \cap I(\mu_{I,x^*}(x_N), \mu_{I,x^*}(y, x_{N \smallsetminus \{i\}})) \cap I(x_i, \mu_{I,x^*}(y, x_{N \smallsetminus \{i\}}))$ whence

$\mu_{I,x^*}(x_N) = m \in I(x_i, \mu_{I,x^*}(y, x_{N \smallsetminus \{i\}}))$.

Now,

$\mu_{I,x^*}(x_N) = \vee_{S \subseteq N, |S|=k+1} \wedge_{h \in S} x_h =$

$= (\vee_{S \subseteq N \smallsetminus \{i\}, |S|=k+1, i \in S} \wedge_{h \in S} x_h) \vee (x_i \wedge \vee_{T \subseteq N, |T|=k+1, i \in T} \wedge_{h \in T \smallsetminus \{i\}} x_h)$

while $\mu_{I,x^*}(x_i, \mu_{I,x^*}(x_N), \mu_{I,x^*}(y, x_{N \smallsetminus \{i\}})) = (x_i \wedge ((\vee_{S \subseteq N, |S|=k+1, i \notin S} \wedge_{h \in S} x_h) \vee$

$\vee(\vee_{T \subseteq N, |T|=k+1, i \in T}(x_i \wedge (\wedge_{h \in T \smallsetminus \{i\}} x_h))) \vee$

$\vee((\vee_{S \subseteq N, |S|=k+1, i \notin S} \wedge_{h \in S} x_h) \vee (\vee_{T \subseteq N, |T|=k+1, i \in T}(x_i \wedge (\wedge_{h \in T \smallsetminus \{i\}} x_h))) \wedge ((\vee_{S \subseteq N, |S|=k+1, i \notin S} \wedge_{h \in S} x_h) \vee$

$(\vee_{T \subseteq N, |T|=k+1, i \in T}(y \wedge (\wedge_{h \in T \smallsetminus \{i\}} x_h))) \vee$

$\vee(x_i \wedge ((\vee_{S \subseteq N, |S|=k+1, i \notin S} \wedge_{h \in S} x_h) \vee (\vee_{T \subseteq N, |T|=k+1, i \in T}(y \wedge (\wedge_{h \in T \smallsetminus \{i\}} x_h)))$.

Clearly,

$\vee_{S \subseteq N, |S|=k+1} \wedge_{h \in S} x_h = (\vee_{S \subseteq N, |S|=k+1, i \notin S} \wedge_{h \in S} x_h) \vee (\vee_{T \subseteq N, |T|=k+1, i \in T}(\wedge_{h \in T} x_h))$,

$(x_i \wedge ((\vee_{S \subseteq N, |S|=k+1, i \notin S} \wedge_{h \in S} x_h) \vee (\vee_{T \subseteq N, |T|=k+1, i \in T}(x_i \wedge (\wedge_{h \in T \smallsetminus \{i\}} x_h))) =$

$(x_i \wedge (\vee_{S \subseteq N, |S|=k+1, i \notin S} \wedge_{h \in S} x_h)) \vee (\vee_{T \subseteq N, |T|=k+1, i \in T} \wedge_{h \in T} x_h))$,

$(\vee_{S \subseteq N, |S|=k+1, i \notin S} \wedge_{h \in S} x_h) \vee (\vee_{T \subseteq N, |T|=k+1, i \in T}(x_i \wedge (\wedge_{h \in T \smallsetminus \{i\}} x_h))) \wedge$

$\wedge((\vee_{S \subseteq N, |S|=k+1, i \notin S} \wedge_{h \in S} x_h) \vee (\vee_{T \subseteq N, |T|=k+1, i \in T}(y \wedge (\wedge_{h \in T \smallsetminus \{i\}} x_h))) =$

$= (\vee_{S \subseteq N, |S|=k+1, i \notin S} \wedge_{h \in S} x_h) \vee$

$\vee(\vee_{T \subseteq N, |T|=k+1, i \in T}(x_i \wedge (\wedge_{h \in T \smallsetminus \{i\}} x_h)) \wedge (\vee_{T \subseteq N, |T|=k+1, i \in T}(y \wedge (\wedge_{h \in T \smallsetminus \{i\}} x_h)))$,

$(x_i \wedge ((\vee_{S \subseteq N, |S|=k+1, i \notin S} \wedge_{h \in S} x_h) \vee (\vee_{T \subseteq N, |T|=k+1, i \in T}(y \wedge (\wedge_{h \in T \smallsetminus \{i\}} x_h))) \leqslant$

$\leqslant (x_i \wedge (\vee_{S \subseteq N, |S|=k+1, i \notin S} \wedge_{h \in S} x_h)) \vee (\vee_{T \subseteq N, |T|=k+1, i \in T} \wedge_{h \in T} x_h))$,

whence

$\mu_I(x_i, \mu_{I,x^*}(x_N), \mu_{I,x^*}(y, x_{N \setminus \{i\}})) =$
$= (x_i \wedge (\vee_{S \subseteq N, |S|=k+1, i \notin S} \wedge_{h \in S} x_h)) \vee (\vee_{T \subseteq N, |T|=k+1, i \in T} \wedge_{h \in T} x_h)) =$
$= \vee_{S \subseteq N, |S|=k+1} \wedge_{h \in S} x_h$
as required. ■

Lemma 8 *Let $\mathcal{I} = (X, I)$ be a median interval space, $N = \{1, ..., 2k+1\}$ for some positive integer k, $x^* \in X$ and $\mu_{x^*} : X^N \longrightarrow X$ the n-ary median rule on X. Then, μ_{x^*} is coalitionally strategy-proof on $U_{\mathcal{I}}^N$ only if it is also a CW-selection on $U_{\mathcal{I}}^N$.*

Proof. Indeed, suppose that on the contrary μ_{x^*} is coalitionally strategy-proof on on $U_{\mathcal{I}}^N$ but is *not* a CW-selection on $U_{\mathcal{I}}^N$. Then, there exist $y \in X$, $\succcurlyeq_N \in U_{\mathcal{I}}^N$ and $x_N \in X^N$ such that $x_i = top(\succcurlyeq_i)$ for all $i \in N$, $\mu_{x^*}(x_N) = z$ and $n_{yz}((\succcurlyeq_i)_{i \in N}) = |\{i \in N : y \succcurlyeq_i z\}| > |N|/2$ hence $n_{zy}((\succcurlyeq_i)_{i \in N}) = |\{i \in N : z \succ_i y\}| < |N|/2$. Now, posit $S = \{i \in N : y \succ_i z\}$ and $y_S = (y_i = y)_{i \in S}$. Thus, by definition, $\mu_{x^*}(y_S, x_{N \setminus S}) = y$ which is therefore manipulable by S at $(\succcurlyeq_i)_{i \in N}$ i.e. is not strategy-proof on $U_{\mathcal{I}}^N$, a contradiction. ■

Theorem 9 *Let $\mathcal{I} = (X, I)$ be a median interval space. Then,*

(i) for any positive integer k, $N = \{1, 2, ..., 2k+1\}$, and $x^* \in X$, if $\mathcal{I}$ satisfies interval anti-exchange then the (extended) median rule $\mu_{I,x^*} : X^N \longrightarrow X$ is a CW-selection on $U_{\mathcal{I}}^N$;

(ii) for any positive integer k, $N = \{1, 2, ..., 2k+1\}$ and $x^* \in X$, if $\mathcal{I}$ is discrete and the (extended) median rule $\mu_{I,x^*} : X^N \longrightarrow X$ is a CW-selection on $\mathcal{U}_{\mathcal{I}}^N$ then the μ_I-induced graph $G_{\mu_I} = (X, E_{\mu_I})$ is a tree.

Proof. (i) Let $\mathcal{I} = (X, I)$ be an interval space that satisfies interval anti-exchange. By Lemma 6, for any N of odd size and any $x^* \in X$, the extended median $\mu_{I,x^*} : X^N \longrightarrow X$ is $\mathcal{I}$-monotonic, hence strategy-proof on $U_{\mathcal{I}}^N$ by Proposition 4. Therefore, by Theorem 5 above, μ_{I,x^*} is also coalitional strategy-proof on $U_{\mathcal{I}}^N$. It follows that, by Lemma 7 above, μ_{I,x^*} is a CW-selection on $U_{\mathcal{I}}^N$.

(ii) To begin with, recall that -for all $x, y \in X$- $(x, y) \in E_{\mu_I}$ iff $I_{\mu_I}(x, y) := I(x, y) = \{x, y\}$ by definition, and observe that by Proposition 1 (ii), $\mu_{I,x^*} = \mu_I$ for all $x^* \in X$. Also, notice that if μ_{I,x^*} is a CW-selection on $U_{\mathcal{I}}^N$ then $\mathcal{I}$-induced graph $G_{\mu_I} = (X, E_{\mu_I})$ must be square-free: to see this, suppose that it is *not* i.e. there exist four distinct elements $x, y, v, z \in X$ such that $E_{\mu_I} =$

$\{\{x,y\},\{y,z\},\{z,v\},\{v,x\}\}$ hence $I_{\mu_I}(x,y)=\{x,y\}$, $I_{\mu_I}(x,v)=\{x,v\}$, $I_{\mu_I}(y,z)=\{y,z\}$, $I_{\mu_I}(v,z)=\{v,z\}$. Also, $I_{\mu_I}(x,z)=I_{\mu_I}(y,v)=\{x,y,v,z\}$ (because it must be the case that $\{x,z\}\subset I_{\mu_I}(x,z)$ and $\{y,v\}\subset I_{\mu_I}(y,v)$, but as it easily checked $\mathcal{I}$ would not be median in case $|I_{\mu_I}(x,z)|=3$ or $|I_{\mu_I}(y,v)|=3$). Next, consider total preorder profile $(\succcurlyeq^*,\succcurlyeq',\succcurlyeq^\circ)$ such that $v\succ^* z\succ^* x\sim^* y\sim^* u$, $\quad y\succ' z\succ' x\sim' v\sim' u$, and $x\succ^\circ y\sim^\circ v\sim^\circ z\sim^\circ u$ for all $u\in X$. By construction, $\mathcal{I}$-unimodality of $\succcurlyeq^*$and $\succcurlyeq'$, and $\succcurlyeq^\circ$only requires that $[\ a\notin I_{\mu_I}(v,z)$ for all $a\in X\smallsetminus\{v,z\}]$, $[\ a\notin I_{\mu_I}(y,z)$ for all $a\in X\smallsetminus\{y,z\}]$, respectively, while $\succcurlyeq^\circ$is unimodal for *any* choice of an interval function on X. Therefore, $(\succcurlyeq^*,\succcurlyeq',\succcurlyeq^\circ)\in U^3_{\mathcal{I}}$ i.e. is unimodal. Now, the median $\mu_I:X^3\longrightarrow X$ as defined by the rule $\mu_I(z_1,z_2,z_3)=m$ such that $\{m\}=I_{\mu_I}(z_1,z_2)\cap I_{\mu_I}(z_2,z_3)\cap I_{\mu_I}(z_1,z_3)$ for any $z_1,z_2,z_3\in X$ (according to Proposition 1 (i)) is *not* a CW-selection on $U^3_{\mathcal{I}}$ since $I_{\mu_I}(v,y)\cap I_{\mu_I}(y,x)\cap I_{\mu_I}(v,x)=\{x,y,v,z\}\cap\{x,y\}\cap\{x,v\}=\{x\}$ hence $\mu(top(\succcurlyeq^*),top(\succcurlyeq'),top(\succcurlyeq^\circ))=\mu(v,y,x)=x$ but $n_{zx}((\succcurlyeq^*,\succcurlyeq',\succcurlyeq^\circ))=2$ while $n_{xz}((\succcurlyeq^*,\succcurlyeq',\succcurlyeq^\circ))=1$. Furthermore, observe that $\mathcal{I}=(X,I)$ is a discrete median interval space implies - by Proposition 1 (ii) and Proposition 2 above- that $G_{\mu_I}=(X,E_{\mu_I})$ is a *median graph* hence cannot include any odd cycle (indeed, if $(x_0,x_1,...,x_{2k},x_0)$ then $I_{G_{\mu_I}}(x_k,x_{k+1})=\{x_k,x_{k+1}\}$, $x_k\notin I_{G_{\mu_I}}(x_0,x_{k+1})$ and $x_{k+1}\notin I_{G_{\mu_I}}(x_0,x_k)$ whence $I_{G_{\mu_I}}(x_k,x_{k+1})\cap I_{G_{\mu_I}}(x_0,x_{k+1})\cap I_{G_{\mu_I}}(x_0,x_k)=\varnothing$). Thus, the shortest cycle in $G_{\mu_I}=(X,E_{I,x})$ has to be of length $h=2k$ with $k\geq 3$. Let then $(x_1,...,x_{2k},x_1)$ be one such cycle, and consider the triple (x_1,x_3,x_{k+2}): by construction, $I_{G_{\mu_I}}(x_1,x_3)=\{x_1,x_2,x_3\}$, $I_{G_{\mu_I}}(x_3,x_{k+2})=\{x_3,...,x_{k+2}\}$, and -since $2k-(k+2)+1=k-1$-

$I_{G_{\mu_I}}(x_1,x_{k+2})=\{x_{k+2},x_{k+3},...,x_{2k},x_1\}$, whence $I_{G_{\mu_I}}(x_1,x_{k+2})\cap I_{G_{\mu_I}}(x_1,x_3)\cap I_{G_{\mu_I}}(x_3,x_{k+2})=\varnothing$, a contradiction again. It follows that G_{μ_I} is acyclic as claimed. But, since $\mathcal{I}$ is discrete, G_{μ_I} is also connected, hence is indeed a tree. ■

Hence, in particular, the median rule is *not* a CW-selection on $U^N_{\mathcal{I}}$ when $G_{\mu_I}=(X,E_{\mu_I})$ is a square (i.e. a finite, regular, median graph of order four). That result is to be contrasted with Proposition 9 of Bandelt, Barthélemy (1984) which establishes that the median rule selects the strict Condorcet winner on any profile of *metric* unimodal total preorders of a median interval space (X,I) if and only if $G_{\mu_I}=(X,E_{\mu_I})$ is cube-free (namely does not include a cube, i.e. a finite, regular, median graph of order eight). Notice however that a metric unimodal total preorder on a square cannot be (incidence) unimodal, and conversely an (incidence) unimodal total preorder cannot be metric unimodal. Another implication of Theorem 9 is that the (extended) median rule invariably selects a Condorcet winner on an odd (incidence) unimodal domain $U_{\mathcal{I}}^N$ whenever $\mathcal{I} = (X, I)$ is the interval space canonically induced by a tree, as made precise by the following Lemma and Corollary.

Lemma 10 *Let $G = (X, E)$ be a tree and $\mathcal{I}_G = (X, I_G)$ the interval space induced by G. Then $\mathcal{I}_G$ is median and satisfies interval anti-exchange.*

Proof. That $\mathcal{I}_G = (X, I_G)$ is median whenever G is a tree is a well-known fact: let us provide an informal sketch of the proof for the sake of completeness. Take any $x, y, z \in X$, and consider any pair of them, say x and y. By definition of tree there exists a unique path from x to y. If z belongs to that path we are done, because by construction $z \in I_G(x, y) \cap I_G(y, z) \cap I_G(x, z)$ and any other $v \in I_G(x, y)$ would be either in $I_G(y, z)$ or in $I_G(x, z)$ but could not possibly be in $I_G(y, z) \cap I_G(x, z)$. By contrast, if z does not belong to the unique path joining x and y, consider the unique path joining z and one of the other two vertices, say x: y either lies on that path or it doesn't. If it does, then clearly $y \in I_G(x, y) \cap I_G(y, z) \cap I_G(x, z)$, and any other vertex $v \in I_G(x, z)$, again, will belong to either $I_G(x, y)$ or to $I_G(y, z)$ but not to $I_G(x, y) \cap I_G(y, z)$. If it doesn't, then $x \in I_G(x, y) \cap I_G(y, z) \cap I_G(x, z)$,and any other vertex $v \in I_G(x, z)$, will not belong to $I_G(x, y)$. Thus, in any case $|I_G(x, y) \cap I_G(y, z) \cap I_G(x, z)| = 1$ i.e. (X, I_G) is median.

Concerning interval anti-exchange, let $x, y \in X$, $x \neq y$, such that $x \in I_G(y, v)$ and $y \in I_G(x, z)$ namely

$d_G(y, v) = d_G(y, x) + d_G(x, v)$ and $d_G(x, z) = d_G(x, y) + d_G(y, z)$.

Thus, $d_G(x, v) = d_G(y, v) - d_G(x, y)$, whence $d_G(z, x) + d_G(x, v) = d_G(x, y) + d_G(y, z) + d_G(y, v) - d_G(x, y) = d_G(z, y) + d_G(y, v)$.

But then, since x is on the unique path joining y and v, and y is on the unique path joining x and z, there exists a path through y joining z and v, whence $d_G(z, v) = d_G(z, y) + d_G(y, v)$. It follows that $d_G(z, v) = d_G(z, x) + d_G(x, v)$, i.e. $x \in I_G(v, z)$ as required. ■

Corollary 11 *Let $G = (X, E)$ be a tree and $\mathcal{I}_G = (X, I_G)$ the interval space induced by G, $N = \{1, 2, ..., 2k + 1\}$ for some positive integer k, and $x \in X$. Then, the median rule $\mu_x : X^N \longrightarrow X$ is a CW-selection on $U_{\mathcal{I}_G}^N$.*

Proof. Immediate from Theorem 9 and Lemma 10. ■

Remark Notice that the foregoing Corollary is essentially stronger than Theorem 4 in Wendell, McKelvey (1981) which establishes that the median is the unique strict Condorcet winner on the set of *metric* unimodal profiles of an odd population of voters if $G = (X, E)$ is a *finite* tree: indeed, finiteness of G is not required, and metric-unimodal total preorders on trees are a proper subclass of incidence-unimodal total preorders.

4 Concluding remarks

Versions of the median voter theorem of the 'committee'-variety for *metric unimodal* preference domains are known to hold when the underlying outcome set is a finite tree (see Wendell, McKelvey (1981)), and, more generally, if and only if it is cube-free i.e. does not include a cube (see Bandelt, Barthélemy (1984)): the latter condition clearly implies that the median voter theorem holds for metric unimodal domains over a square. Those facts are to be contrasted with our results concerning the median voter theorem for incidence-unimodal domains of total preorders as summarized by Theorem 9: indeed, Theorem 9 (i) implies that the median voter theorem holds whenever the outcome space is any (possibly infinite) tree or any median geometry that satisfies the interval anti-exchange property as discussed above (a generalization of both Moulin (1980) and Danilov (1994)). On the other hand, Theorem 9 (ii) establishes that the median voter theorem fails for incidence-unimodal domains of total preorders over a square and more generally whenever the network structure induced by the interval space of the outcome set is discrete but not a tree.

References

[1] Avann S.P. (1948): Ternary distributive semi-lattices, *Bulletin of the American Mathematical Society* 54, 79.

[2] Avann S.P. (1961): Metric ternary distributive semi-lattices, *Proceedings of the American Mathematical Society* 12, 401-414.

[3] Bandelt H.J. (1985): Networks with Condorcet solutions, *European Journal of Operational Research* 20, 314-326.

[4] Bandelt H.J., J.P. Barthélemy (1984): Medians in median graphs, *Discrete Applied Mathematics* 8, 131-142.

[5] Bandelt H.J., J. Hedlíková (1983): Median algebras, *Discrete Mathematics* 45, 1-30.

[6] Birkhoff G., S.A. Kiss (1947): A ternary operation in distributive lattices, *Bulletin of the American Mathematical Society* 53, 749-752.

[7] Black D. (1948): On the rationale of group decision-making, *Journal of Political Economy* 56, 23-34.

[8] Coppel W.A. (1998): *Foundations of convex geometry.* Cambridge University Press, Cambridge UK.

[9] Danilov V.I. (1994): The structure of non-manipulable social choice rules on a tree, *Mathematical Social Sciences* 27, 123-131.

[10] Downs A. (1957): *An economic theory of democracy*, Harper, New York.

[11] Hansen P., J.F. Thisse (1981): Outcomes of voting and planning. Condorcet, Weber and Rawls locations, *Journal of Public Economics* 16, 1-15.

[12] Hotelling H. (1929): Stability in competition, *Economic Journal* 39, 41-57

[13] Moulin H. (1980): On strategy-proofness and single peakedness, *Public Choice* 35, 437-455.

[14] Moulin H. (1983): *The strategy of social choice.* North Holland, Amsterdam.

[15] Mulder H.M. (1980): *The interval function of a graph.* Mathematical Centre Tracts 132, Amsterdam.

[16] Roemer J.E. (2001): *Political competition. Theory and applications.* Harvard University Press, Cambridge MASS.

[17] Sholander M. (1952): Trees, lattices, order, and betweenness, *Proceedings of the American Mathematical Society* 3, 369-381.

[18] Sholander M. (1954 (a)): Medians and betweenness, *Proceedings of the American Mathematical Society* 5, 801-807.

[19] Sholander M (1954 (b)): Medians, lattices, and trees, *Proceedings of the American Mathematical Society* 5, 808-812.

[20] van de Vel M.L.J. (1993): *Theory of convex structures*. North Holland, Amsterdam.

[21] Vannucci S. (2013): Unimodal domains, anti-exchange properties, and coalitional strategy-proofness of voting rules, Working Paper, University of Siena.

[22] Wendell R.E., R.D. McKelvey (1981): New perspectives in competitive location theory, *European Journal of Operational Research* 6, 174-181.