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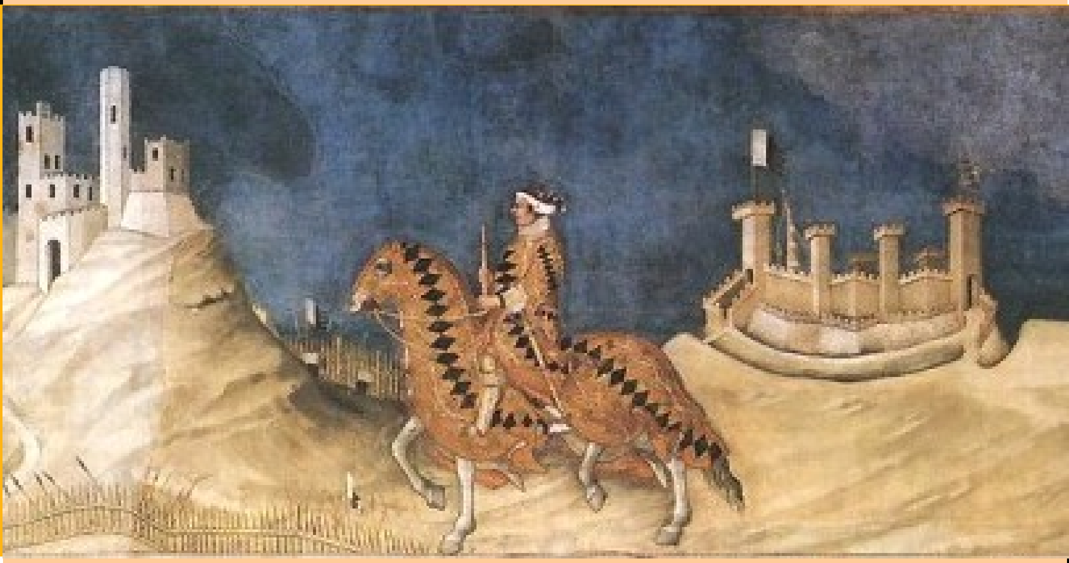

On two-valued nonsovereign strategy-proof voting rules

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Abstract

It is shown that a two-valued and nonsovereign voting rule is strategy-proof on any preference domain that includes all profiles of total preorders with a unique maximum if and only if votes for noneligible feasible alternatives are only available to dummy voters.

It follows that dummy-free two-valued nonsovereign strategy-proof voting rules with a suitably restricted ballot domain do exist and essentially correspond to dummy-free sovereign strategy-proof voting rules for binary outcome spaces or, equivalently, to ordered transversal pairs of order filters of the coalition poset, and are also coalitionally strategy-proof. Moreover, it turns out that two-valued nonsovereign strategy-proof voting rules with full ballot domain do not exist.

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1 Introduction

Good public decisions rely on information that is usually distributed among several stakeholders, and *private*. Thus, voting by a committee is a very effective way to elicit and amalgamate that information provided the underlying voting rule is *strategy-proof*, namely resistant to disruptive attempts at manipulation on the part of individual voters (and *coalitionally strategy-proof* as well if voters can communicate). But not all strategy-proof voting rules are acceptable. In principle, every stakeholder should have a say on the final decision, and every feasible alternative should have a chance to prevail when supported by a suitably large coalition of voters. Therefore, each voter should be able to influence the final outcome under certain profiles of votes, and each feasible outcome should be actually eligible under some distributions of votes: in other terms, a nice strategy-proof voting rule should be *fully polyarchic* -or *dummy-free*- and *sovereign*. Of course, those requirements rule out -in particular- projections or dictatorial rules and constant rules that are indeed coalitionally strategy-proof but also clearly violate dummy-freeness, and both dummy-freeness and sovereignty, respectively.

It is well-known that if there are only *two feasible outcomes* then dummy-freeness and sovereignty are in fact mutually consistent properties for some strategy-proof voting rules. Even with just two voters duple asymmetric-veto voting rules (requiring unanimity to choose one of the two alternatives and just one favourable vote to choose the other one) are indeed dummy-free, sovereign and coalitionally strategy-proof. Moreover, with three or more voters, several variants of the simple majority rule -that are indeed dummy-free, sovereign and coalitionally strategy-proof- are also available.

With *three or more feasible outcomes*, however, the prospects for nice strategy-proof voting rules look definitely much less promising. In fact, the well-known Gibbard-Satterthwaite theorem implies that in that case dummy-freeness and sovereignty (or indeed even just allowance for at least three distinct eligible outcomes within its range) are inconsistent properties for any strategy-proof voting rule unless the admissible preference preorders of voters are suitably restricted (see e.g. Danilov, Sotskov (2002), Taylor (2005)). Thus, with three or more feasible outcomes and no demanding restrictions on admissible preference rankings, one should rather take into consideration some weaker combination of the former polyarchy-cum-sovereignty requirements. In that connection, it seems to be quite natural to focus on two minimal versions of the foregoing polyarchy and sovereignty properties, namely *'minimal polyarchy'* ('there exist at least *two* nondummy voters') and *'minimal sovereignty'* ('the range of the voting rule comprises at least *two* distinct outcomes').

Therefore, one might consider the prospects for either 'minimal polyarchy' and sovereignty, or dummy-freeness and 'minimal sovereignty' of a strategy-proof voting rule. Unfortunately, as it is immediately apparent, the Gibbard-Satterthwaite theorem also rules out the existence of strategy-proof voting rules that satisfy both 'minimal polyarchy' and sovereignity, by establishing that only dictatorships are strategy-proof among the voting rules admitting three or more distinct outcomes in their range: arguably, that is one very good reason to regard the Gibbard-Sattertshwaite as an 'impossibility theorem', precisely as it has been always done in the social choice and voting literature. Thus, it only remains to be explored the other possibility, namely the existence of *dummy-free and 'minimally sovereign' strategy-proof* voting rules or equivalently -in view of the Gibbard-Satterthwaite theorem- of two-valued (hence in particular nonsovereign) dummy-free strategy-proof voting rules. A similar analysis of two-valued nonsovereign strategy-proof *social choice functions* with arbitrary domains of profiles of total preorders has been recently produced by Barberà, Berga, Moreno (2012) (but see also Larsson, Svennson (2006) and Manjunath (2009)), implying that both duple asymmetric-veto and duple serial dictatorships provide examples of nonsovereign dummy-free and coalitionally strategy-proof social choice functions. Relying on a much more parsimonious information base, voting rules may be regarded as a proper subclass of social choice functions, but at the same time they also offer both less available strategies and more opportunities for manipulation. Thus, a special analysis is required for a characterization of two-valued nonsovereign, dummy-free, and strategy-proof voting rules having arbitrary ballot domains i.e. *arbitrary subspaces of the outcome set as strategy spaces.* The present note is aimed at filling this small but significant gap in the literature. We shall prove that under suitably restricted ballot domains such voting rules do indeed exist, but essentially coincide with dummy-free sovereign and strategy-proof voting rules for *binary* outcome sets, including simple majority-based rules. Furthermore, two-valued nonsovereign strategy-proof voting rules are *in one-to-one correspondence with ordered transversal pairs of order filters of the coalition poset* (denoting the families of weakly decisive coalitions for the two eligible outcomes). It will also be shown that, by contrast, two-valued nonsovereign strategy-proof voting rules *with full ballot domain* -i.e. allowing votes over the entire outcome space- *do not exist.*

2 Two-valued nonsovereign strategy-proof voting rules

Let $N = \{1, ..., n\}$ denote the population of voters and X the set of alternative outcomes: we assume $|N| \geq 2$ and $|X| \geq 3$. Thus, the power set $\mathcal{P}(N)$ of N denotes the set of all possible coalitions, and partially ordered set $(\mathcal{P}(N), \subseteq)$ denotes the coalition poset. A family $\mathcal{W} \subseteq \mathcal{P}(N)$ of coalitions is an order filter of $(\mathcal{P}(N), \subseteq)$ iff for any $S, T \subseteq N$, if $S \in \mathcal{W}$ and $S \subseteq T$ then $T \in \mathcal{W}$. For any $Y \subseteq X$, let $\succcurlyeq$ denote a total preorder i.e. a reflexive, connected and transitive binary relation on Y (we shall denote by $\succ$ and $\sim$ its asymmetric and symmetric components, respectively). We denote by U_Y the set of all total preorders on Y. For any $(Y_i \subseteq X)_{i \in N}$, a N-profile on $(U_{Y_i})_{i \in N}$ is a function mapping each $i \in N$ into $\succcurlyeq_i \in U_{Y_i}$, and $\Pi_{i \in N} U_{Y_i}$ the set of all such profiles, or *universal preference domain* of total preorders on Y_i, $i = 1, ..., n$. We also assume that each $Y \subseteq X$ is endowed with an *interval function* $I : Y^2 \rightarrow \mathcal{P}(Y)$ such that $\mathcal{I}_Y = (Y, I_Y)$ is an **interval space** i.e. I_Y satisfies the following two conditions:

I-(i) (**Extension**): $\{x, y\} \subseteq I_Y(x, y)$ for all $x, y \in Y$,

I-(ii) (**Symmetry**): $I_Y(x, y) = I_Y(y, x)$ for all $x, y \in Y$.

In particular, we also assume that $n \geq 2$ in order to avoid tedious qualifications, and that $\mathcal{I}_Y = (Y, I_Y)$ is an *idempotent* interval space namely that

(**Idempotence**): $I_Y(x, x) = \{x\}$ for all $x \in X$

is also satisfied.

A subset $Z \subseteq Y \subseteq X$ is $\mathcal{I}_Y$-*convex* iff $I_Y(x, y) \subseteq Z$ for all $x, y \in Z$. For any $Z \subseteq Y$, the $\mathcal{I}_Y$-*convex hull* of Z - denoted $co_{\mathcal{I}_Y}(Z)$- is the smallest $\mathcal{I}_Y$-convex superset of Z, namely $co_{\mathcal{I}_Y}(Z) = \bigcap \{A \subseteq Y : A \text{ is } \mathcal{I}_Y\text{-}convex \text{ and } A \supseteq Z\}$.

An interval space $\mathcal{I}_Y = (Y, I_Y)$ is *convex* if I_Y also satisfies

(**Convexity**): $I_Y(x, y)$ is $\mathcal{I}_Y$-convex for all $x, y \in Y$.

Then, $\succcurlyeq$ is said to be *unimodal* with respect to interval space $\mathcal{I}_Y = (Y, I_Y)$ - or $\mathcal{I}_Y$-**unimodal** - if and only if

U-(i) there exists a *unique maximum* of $\succcurlyeq$ in Y, its *top* outcome -denoted *top*$(\succcurlyeq)$- and

U-(ii) for all $x, y, z \in X$, if $z \in I_Y(x, y)$ then $\{u \in Y : z \succcurlyeq u\} \cap \{x, y\} \neq \varnothing$.

We denote by $U_{\mathcal{I}_Y}$ the set of all $\mathcal{I}_Y$-*unimodal* total preorders on Y. For any $(Y_i \subseteq X)_{i\in N}$, a N-profile on $(U_{\mathcal{I}_{Y_i}})_{i\in N}$ is a function mapping each $i \in N$ into $\succcurlyeq_i \in U_{\mathcal{I}_{Y_i}}$, and $\Pi_{i\in N} U_{\mathcal{I}_{Y_i}}$ denotes the set of all N-profiles of $\mathcal{I}_{Y_i}$-unimodal total preorders.

In particular, we shall focus on the clique-induced interval space $\mathcal{I}_Y^{\mathcal{C}}=(Y, I_Y^{\mathcal{C}})$ that is defined as follows: for any $x, y \in Y$, $I_Y^{\mathcal{C}}(x,y) = \{x,y\}$; notice that $\mathcal{I}_Y^{\mathcal{C}}=(Y, I^{\mathcal{C}})$ is by construction both idempotent and convex. Notice that the set of all $\mathcal{I}_X^{\mathcal{C}}$-unimodal total preorders is the set of all total preorders on X with a unique maximum.

A **voting rule** for (N,X) with ballot domain $\mathbf{D}$, is a function $f : \mathbf{D} \to X$ where $\mathbf{D} = \Pi_{i\in N} Y_i$ for some $(Y_i : Y_i \subseteq X)_{i\in N}$. A voter $i \in N$ is a *dummy* for voting rule f iff $f(y_i, (z_j)_{j\in N\smallsetminus\{i\}}) = f(z_i, (z_j)_{j\in N\smallsetminus\{i\}})$ for all $y_i, z_i \in Y_i$ and all $(z_j)_{j\in N\smallsetminus\{i\}} \in \Pi_{j\in N\smallsetminus\{i\}} Y_j$. We denote by $D_f \subseteq N$ the subset of *dummy voters* for voting rule f. A voting rule is *trivial* if $D_f = N$, and *dummy-free* if $D_f = \varnothing$. For all $x \in f[\mathbf{D}]$, we also posit

$$\mathcal{W}_x^f = \left\{ \begin{array}{c} S \subseteq N : \text{ if } y_i = x \text{ for each } i \in S \text{ and} \\ y_j \neq x \text{ for each } j \in N \smallsetminus S \\ \text{then } f((y_i)_{i\in N}) = x \end{array} \right\}:$$

$\mathcal{W}_x^f$ denotes the family of coalitions that are *weakly decisive* for x under voting rule f.

Clearly, for any two *distinct* $x, y \in f[\mathbf{D}]$, it must be the case that $S \cap T \neq \varnothing$ for all $S \in \mathcal{W}_x^f$, $T \in \mathcal{W}_y^f$ i.e. $(\mathcal{W}_x^f, \mathcal{W}_y^f)$ is a *transversal pair* of families of coalitions.

A voting rule $f : \mathbf{D} \to X$ is **strategy-proof** *on preference domain* $V^N \subseteq U^N_{\Pi_i Y_i}$ iff for all N-profiles $(\succcurlyeq_i)_{i\in N} \in V^N$, and for all $i \in N$, $y_i \in Y_i$, and $(x_j)_{j\in S} \in \mathbf{D}$ such that $(y_i, (x_j)_{j\in S\smallsetminus\{i\}}) \in \mathbf{D}$, and $x_j \succcurlyeq_j z_j$ for each $j \in N$ and each $z_j \in Y_j$, $f((x_j)_{j\in N}) \succcurlyeq_i f((y_i, (x_j)_{j\in N\smallsetminus\{i\}}))$ holds. Moreover, a voting rule f is **coalitionally strategy-proof** *on* V^N iff for all N-profiles $(\succcurlyeq_i)_{i\in N} \in V^N$, $(x_j)_{j\in N} \in \Pi_{i\in N} Y_i$, $S \subseteq N$ and $(y_i)_{i\in S} \in \Pi_{i\in S} Y_i$ such that $x_j \succcurlyeq_j z_j$ for each $j \in N$ and each $z_j \in Y_j$, there exists $i \in S$ with $f((x_j)_{j\in N}) \succcurlyeq_i f((y_i)_{i\in S}, (x_j)_{j\in N\smallsetminus S}))$.

Finally, a voting rule $f : \mathbf{D} \to X$ with $\mathbf{D} = \Pi_{i\in N} Y_i$ is $\mathcal{I}_{\mathbf{D}}$**-monotonic** iff for all $i \in N$, $y_i \in Y_i$ and $(x_j)_{j\in N} \in \mathbf{D}$, $f((x_j)_{j\in N}) \in I_{Y_i}(x_i, f(y_i, (x_j)_{j\in N\smallsetminus\{i\}}))$.

We are now ready to state the main results of this paper. Our results rely on the following lemma that establishes the equivalence between *monotonicity* of a voting rule $f : \mathbf{D} \to X$ with respect to an arbitrary profile of convex idempotent interval spaces $(\mathcal{I}_{Y_i})_{i\in N}$ and *strategy-proofness on the corresponding (full) unimodal preference domain* $\Pi_{i\in N} U_{\mathcal{I}_{Y_i}}$ (the first equivalence established by that Lemma is an extension of a similar result for the interval spaces of trees due to Danilov (1994)).

Lemma 1 *Let* $Y_i \subseteq X$*, and* $\mathcal{I}_{Y_i} = (Y_i, I_{Y_i})$ *a convex idempotent interval space for any* $i \in N$*, and* $f : \mathbf{D} \to X$ *a two-valued voting rule with* $\mathbf{D} = \Pi_{i \in N} Y_i$ *and* $|X| \geq 3$*. Then, the following statements are equivalent:*

(i) f *is strategy-proof on its full unimodal preference domain* $\Pi_{i \in N} U_{\mathcal{I}_{Y_i}}$ *;*

(ii) f *is* $\mathcal{I}_{\mathbf{D}}$*-monotonic;*

(iii) f *is* $\mathcal{I}_{\mathbf{D}}$*-monotonic and* $\mathcal{W}_x^f$ *is an order filter of* $(\mathcal{P}(N), \subseteq)$ *for all* $x \in f[\mathbf{D}]$*.*

Proof. (i) $\Longrightarrow$(ii): Let $f(\mathbf{D}) = \{x, y\}$ and suppose that $f : \mathbf{D} \to X$ is *not* $\mathcal{I}_{\mathbf{D}}$-monotonic: thus, there exist $i \in N$, $x_i' \in Y_i$, and $x_N = (x_i)_{i \in N} \in \mathbf{D}$ such that $f(x_N) \notin I_{Y_i}(x_i, f(x_i', x_{N \smallsetminus \{i\}}))$. Then, consider the total preorder $\succcurlyeq^*$ on Y_i defined as follows: $x_i = top(\succcurlyeq^*)$ and for all $y, z \in Y_i \smallsetminus \{x_i\}$, $y \succcurlyeq^* z$ iff (i) $\{y, z\} \subseteq I_{Y_i}(x_i, f(x_i', x_{N \smallsetminus \{i\}})) \smallsetminus \{x_i\}$ or (ii) $y \in I_{Y_i}(x_i, f(x_i', x_{N \smallsetminus \{i\}})) \smallsetminus \{x_i\}$ and $z \notin I_{Y_i}(x_i, f(x_i', x_{N \smallsetminus \{i\}}))$ or (iii) $y \notin I_{Y_i}(x_i, f(x_i', x_{N \smallsetminus \{i\}}))$ and $z \notin I_{Y_i}(x_i, f(x_i', x_{N \smallsetminus \{i\}}))$. Clearly, by construction $\succcurlyeq^*$consists of three indifference classes with $\{x_i\}$, $I_{Y_i}(x_i, f(x_i', x_{N \smallsetminus \{i\}})) \smallsetminus \{x_i\}$ and $X \smallsetminus I_{Y_i}(x_i, f(x_i', x_{N \smallsetminus \{i\}}))$ as top, medium and bottom indifference classes, respectively.

Now, observe that $\succcurlyeq^* \in U_{\mathcal{I}_{Y_i}}$. To check that statement, take any $y, z, v \in Y_i$ such that $y \neq z$ and $v \in I_{Y_i}(y, z)$ (if $y = z$ then, by Idempotence of $\mathcal{I}_{Y_i}$, $v = y = z$ and there is in fact nothing to prove). Also, notice that $\{y, z\} \neq \{x_i\}$ since $y \neq z$, and assume without loss of generality that $y \neq x_i$.

If $\{y, z\} \subseteq I_{Y_i}(x_i, f(x_i', x_{N \smallsetminus \{i\}}))$ then, by Convexity of $\mathcal{I}_{Y_i}$,
$v \in I_{Y_i}(x_i, f(x_i', x_{N \smallsetminus \{i\}}))$. Hence, $v \succcurlyeq^* y$ by definition of $\succcurlyeq^*$.

If on the contrary $\{y, z\} \cap (Y_i \smallsetminus I_{Y_i}(x_i, f(x_i', x_{N \smallsetminus \{i\}}))) \neq \varnothing$ then take $w \in \{y, z\} \cap (Y_i \smallsetminus I_{Y_i}(x_i, f(x_i', x_{N \smallsetminus \{i\}})))$.

Clearly, by definition of $\succcurlyeq^*$again, $v \succcurlyeq^* w$. Since $w \in \{y, z\}$, it follows that the unimodality condition is satisfied again and therefore $\succcurlyeq^* \in U_{\mathcal{I}_{Y_i}}$ as claimed.

Also, by assumption $f(x_N) \in X \smallsetminus I_{Y_i}(x_i, f(x_i', x_{N \smallsetminus \{i\}}))$ while $f(x_i', x_{N \smallsetminus \{i\}}) \in I_{Y_i}(x_i, f(x_i', x_{N \smallsetminus \{i\}}))$ by Extension, whence by construction $f(x_i', x_{N \smallsetminus \{i\}}) \succ^* f(x_N)$. But then, f is *not* strategy-proof on $\Pi_{i \in N} U_{\mathcal{I}_{Y_i}}$.

(ii) $\Longrightarrow$(i): Conversely, let f be $\mathcal{I}_{\mathbf{D}}$-monotonic. Now, consider any $\succcurlyeq = (\succcurlyeq_j)_{j \in N} \in \Pi_{i \in N} U_{\mathcal{I}_{Y_i}}$ and any $i \in N$. By definition of $\mathcal{I}_{\mathbf{D}}$-monotonicity $f(top(\succcurlyeq_i), x_{N \smallsetminus \{i\}}) \in I_{Y_i}(top(\succcurlyeq_i), f(x_i, x_{N \smallsetminus \{i\}}))$ for all $x_{N \smallsetminus \{i\}} \in \Pi_{j \in N \smallsetminus \{i\}} Y_j$

and $x_i \in Y_i$. But then, since clearly by definition $top(\succcurlyeq_i) \succcurlyeq_i f(top(\succcurlyeq_i), x_{N \smallsetminus \{i\}})$, either $f(top(\succcurlyeq_i), x_{N \smallsetminus \{i\}}) = top(\succcurlyeq_i)$ or $f(top(\succcurlyeq_i), x_{N \smallsetminus \{i\}}) \succcurlyeq_i f(x_i, x_{N \smallsetminus \{i\}})$ by unimodality of $\succcurlyeq_i$. Hence, $f(top(\succcurlyeq_i), x_{N \smallsetminus \{i\}}) \succcurlyeq_i f(x_i, x_{N \smallsetminus \{i\}})$ in any case. It follows that f is indeed strategy-proof on $\Pi_{i \in N} U_{\mathcal{I}_{Y_i}}$.

(iii)$\Longrightarrow$(ii): Trivial.

(ii)$\Longrightarrow$(iii): Indeed, suppose that f is $\mathcal{I}_{\mathbf{D}}$-monotonic and there exist $x \in f[\mathbf{D}]$ and $S \subseteq T \subseteq N$ such that $S \in \mathcal{W}_x^f$ and $T \notin \mathcal{W}_x^f$. Hence, by finiteness of N, there exist S' with $S \subseteq S' \subseteq T$, and $i \in T \smallsetminus S'$ such that $S' \in \mathcal{W}_x^f$ and $S' \cup \{i\} \notin \mathcal{W}_x^f$. Thus, $f((x_j = x)_{j \in S}, (y_j)_{j \in N \smallsetminus S}) = x$ and $f((x_j = x)_{j \in S \cup \{i\}}, (y_i)_{i \in N \smallsetminus (S \cup \{i\})}) \neq x$ hence, by Idempotence of $\mathcal{I}_{Y_i}$, $f((x_j = x)_{j \in S \cup \{i\}}, (y_i)_{i \in N \smallsetminus (S \cup \{i\})}) \notin I_{Y_i}(x, f((x_j = x)_{j \in S}, (y_j)_{j \in N \smallsetminus S})$, which contradicts $\mathcal{I}_{\mathbf{D}}$-monotonicity of f. ■

We are now ready to state our characterization results.

Theorem 2 *Let $f : \mathbf{D} \rightarrow X$ be a two-valued voting rule with $|X| \geq 3$, $\mathcal{I}_{Y_i}^{\mathcal{C}}$=$(Y_i, I_{Y_i}^{\mathcal{C}})$ the clique-induced interval spaces on Y_i, $i = 1, ..., n$, and $\Pi_{i \in N} U_{\mathcal{I}_{Y_i}^{\mathcal{C}}} \subseteq \mathcal{D} \subseteq \Pi_{i \in N} U_{Y_i}$. Then, the following statements are equivalent:*

(i) f is strategy-proof on preference domain $\mathcal{D}$;

(ii) f is $\mathcal{I}_{\mathbf{D}}^{\mathcal{C}}$-monotonic;

(iii) f is $\mathcal{I}_{\mathbf{D}}^{\mathcal{C}}$-monotonic and $\mathbf{D} \subseteq (f[\mathbf{D}])^{N \smallsetminus D_f} \times \Pi_{i \in D_f} Y_i$.

Proof. $(i) \Longrightarrow (ii)$To begin with, notice that by construction $\mathcal{I}_{Y_i}^{\mathcal{C}}$=$(Y_i, I_{Y_i}^{\mathcal{C}})$ is indeed a convex idempotent interval space, for all $i \in N$. Moreover, f is strategy-proof on $\mathcal{D}$ entails that f is in particular strategy-proof on $\Pi_{i \in N} U_{\mathcal{I}_{Y_i}^{\mathcal{C}}}$ and therefore -by Lemma 1 above- f is also $\mathcal{I}_{\mathbf{D}}^{\mathcal{C}}$-monotonic;

$(ii) \Longrightarrow (i)$ If f is $\mathcal{I}_{\mathbf{D}}^{\mathcal{C}}$-monotonic then, by Lemma 1 it is also strategy-proof on $\Pi_{i \in N} U_{\mathcal{I}_{Y_i}^{\mathcal{C}}}$. Now, suppose that f is not strategy-proof on $\mathcal{D}$ i.e. there exist $(\succcurlyeq_j)_{j \in N} \in \mathcal{D}$, $(x_j)_{j \in N} \in \mathbf{D}$, $i \in N$, $y_i \in Y_i$ such that $((x_j)_{j \in N \smallsetminus \{i\}}, y_i) \in \mathbf{D}$, $\succcurlyeq_i \in U_{Y_i}$ such that $x_i \succcurlyeq_i z$ for all $z \in X$, and $f((y_i, (x_j)_{j \in N \smallsetminus \{i\}})) \succ_i f((x_j)_{j \in N}$. Then take $\succcurlyeq_i'$defined as follows: $x_i \succ_i y$ for all $y \in X \smallsetminus \{x_i\}$ and $y \succcurlyeq' z$ iff $y \succcurlyeq z$ for any $y, z \in Y_i \smallsetminus \{x_i\}$. Thus, $\succcurlyeq' \in U_{\mathcal{I}_{Y_i}^{\mathcal{C}}}$, by construction. Since $f((y_i, (x_j)_{j \in N \smallsetminus \{i\}})) \succ_i f((x_j)_{j \in N}$ entails that $f((x_j)_{j \in N} \neq x_i$, it follows that $f((y_i, (x_j)_{j \in N \smallsetminus \{i\}})) \succ_i' f((x_j)_{j \in N}$. Therefore, f is not strategy-proof on $U_{\mathcal{I}_{Y_i}^{\mathcal{C}}}^N$, a contradiction;

$(iii) \Longrightarrow (ii)$ Trivial;

$(ii) \Longrightarrow (iii)$ Let us assume that f is $\mathcal{I}^{\mathcal{C}}_{\mathbf{D}}$-monotonic and there exist a non-dummy voter $i \in N \smallsetminus D_f$ and $v_i \in Y_i \smallsetminus \{x, y\}$ such that $f((v_i, (z_j)_{j \in N \smallsetminus \{i\}})) \neq f((w_i, (z_j)_{j \in N \smallsetminus \{i\}}))$ for some $(z_j)_{j \in N \smallsetminus \{i\}} \in \Pi_{j \in N \smallsetminus \{i\}} Y_j$, and $w_i \in Y_i$.

Let us then suppose, without loss of generality, that $x = f((v_i, (z_j)_{j \in N}) \neq f((w_i, ((z_j)_{j \in N \smallsetminus \{i\}})) = y$: notice that, by construction, both $(v_i, (z_j)_{j \in N \smallsetminus \{i\}})$ and $(w_i, ((z_j)_{j \in N \smallsetminus \{i\}})$ are in $\mathbf{D}$. Next, consider any profile $(\succcurlyeq_j)_{j \in N} \in \Pi_{j \in N} U_{\mathcal{I}^{\mathcal{C}}_{Y_j}}$ such that $v_i \succ_i y \succ_i x$. Then, $f((w_i, (v_j)_{j \in N \smallsetminus \{i\}})) = y \succ_i x = f((v_j)_{j \in N})$ i.e. i can manipulate f at $(v_j)_{j \in N} \in \mathbf{D}$, namely f is not strategy-proof on $\mathcal{D}$ hence by the first part of this proof it is not $\mathcal{I}^{\mathcal{C}}_{\mathbf{D}}$-monotonic, a contradiction. ■

As an immediate consequence of Lemma 1 and Theorem 2 we have the following corollaries:

Corollary 3 *Let $f : \mathbf{D} \rightarrow X$ be a dummy-free two-valued voting rule with $|X| \geq 3$, $\mathcal{I}^{\mathcal{C}}_{Y_i}$=$(Y_i, I^{\mathcal{C}}_{Y_i})$ the clique-induced interval spaces on Y_i, $i = 1, ..., n$, and $\Pi_{i \in N} U_{\mathcal{I}^{\mathcal{C}}_{Y_i}} \subseteq \mathcal{D} \subseteq \Pi_{i \in N} U_{Y_i}$. Then, the following statements are equivalent:*

(i) f is strategy-proof on preference domain $\mathcal{D}$;

(ii) $\mathbf{D} = (f[\mathbf{D}])^N$ and $\mathcal{W}^f_x$ is an order filter of $(\mathcal{P}(N), \subseteq)$, for all $x \in f[\mathbf{D}]$.

Proof. (i)$\Longrightarrow$(ii): If f is strategy proof on $\mathcal{D}$ and dummy-free then by Theorem 2 above it follows immediately that f is $\mathcal{I}^{\mathcal{C}}_{\mathbf{D}}$-monotonic and $\mathbf{D} \subseteq (f[\mathbf{D}])^N$. Also, since $|f(\mathbf{D})| = 2$ dummy-freeness of f also entails that $\mathbf{D} = (f[\mathbf{D}])^N$. Moreover, it follows from Lemma 1 above that $\mathcal{W}^f_x$ is an order filter of $(\mathcal{P}(N), \subseteq)$ for all $x \in f[\mathbf{D}]$.

(ii) $\Longrightarrow$(i): Suppose that $\mathbf{D} = (f[\mathbf{D}])^N$ and $\mathcal{W}^f_x$ is an order filter of $(\mathcal{P}(N), \subseteq)$, for all $x \in f[\mathbf{D}]$. Then, f is $\mathcal{I}^{\mathcal{C}}_{\mathbf{D}}$-monotonic: indeed, suppose it is not. Then there exist $u, v \in f[\mathbf{D}] = \{x, y\}$, $i \in N$ and $(z_j)_{j \in N \smallsetminus \{i\}} \in (f[\mathbf{D}])^{N \smallsetminus \{j\}}$ such that $f(u, (z_j)_{j \in N \smallsetminus \{i\}}) \notin I^{\mathcal{C}}(u, f(v, (z_j)_{j \in N \smallsetminus \{i\}})) = \{u, f(v, (z_j)_{j \in N \smallsetminus \{i\}})\}$.

Therefore, it must be the case that $u \neq v$ and $f(u, (z_j)_{j \in N \smallsetminus \{i\}}) \neq u = f(v, (z_j)_{j \in N \smallsetminus \{i\}})$ whence $S = \{j \in N : z_j = u\} \in \mathcal{W}^f_x$, $i \notin S$, and $S \cup \{i\} \notin \mathcal{W}^f_u$ hence $\mathcal{W}^f_u$ is not an order filter of $(\mathcal{P}(N), \subseteq)$, a contradiction. But then, it follows from Theorem 2 above that f -being $\mathcal{I}^{\mathcal{C}}_{\mathbf{D}}$-monotonic- must also be strategy-proof on $\mathcal{D}$ as required. ■

Corollary 4 *Let* $f : X^N \to X$ *be a two-valued voting rule with full ballot domain,* $|N| \geq 2$*,* $|X| \geq 3$*,* $\mathcal{I}^{\mathcal{C}}=(X, I^{\mathcal{C}})$ *the clique-induced interval space on* X*, and* $U^N_{\mathcal{I}^{\mathcal{C}}_X} \subseteq \mathcal{D} \subseteq U^N_X$*. Then,* f *is not strategy-proof on preference domain* $\mathcal{D}$*.*

Proof. Indeed, suppose f is strategy-proof on $\mathcal{D}$. Then, by Theorem 2 above, $X^N \subseteq (f[X^N])^{N \smallsetminus D_f} \times X^{D_f}$ whence $N \smallsetminus D_f = \varnothing$ i.e. $N = D_f$, a contradiction. ■

Thus, Corollary 3 entails that dummy-free nonsovereign two-valued strategy proof voting rules are in a one-to-one correspondence to *ordered transversal pairs* $(\mathcal{W}^f_x, \mathcal{W}^f_y)$ *of order filters of the coalition poset* $(\mathcal{P}(N), \subseteq)$, and do essentially reduce to the class of $\mathcal{I}^{\mathcal{C}}$-monotonic dummy-free and *sovereign* voting rules on *binary* outcome sets. Therefore, in view of Barberà, Berga, Moreno (2010) and Vannucci (2012), *they are also coalitionally strategy-proof.* In particular, they include *simple majority-based* voting rules, which correspond to transversal pairs of order filters $(\mathcal{W}^f_x, \mathcal{W}^f_y)$ such that for any $u \in \{x, y\}$ and $S \subseteq N$, $S \in \mathcal{W}^f_u$ if $|S| > |N \smallsetminus S|$.

Furthermore, Corollary 4 establishes that if $|N| \geq 2$ and $|X| \geq 3$, then there is no two-valued nonsovereign strategy-proof voting rule that is defined on the full ballot domain X^N. That fact is to be contrasted with the existence of two-valued nonsovereign strategy-proof social choice functions on the full domain of total preorders such as serial dictatorships and duple asymmetric-veto rules (see Barberà, Berga, Moreno (2012)). The reason underlying such remarkable difference in the behaviour of two-valued nonsovereign strategy-proof social choice functions and voting rules may be summarized as follows: the limited information on preferences provided by truthful strategies for voting rules may fail to provide information about true preferences among the eligible outcomes leaving thereby some scope for manipulation.

3 Concluding remarks

We have characterized the class of two-valued nonsovereign strategy-proof voting rules, with a view to explore the existence issue for *dummy-free* rules among them. It has been shown that two-valued nonsovereign and dummy-free strategy-proof voting rules with ballot domains restricted to eligible outcomes do indeed exist, and coincide essentially with dummy-free and *sov-*

ereign strategy-proof voting rules on *binary* outcome sets. That is so because, as it turns out, adding feasible but noneligible outcomes to the strategy space of any nondummy voter provides scope for manipulation on the part of that voter. In particular, due to that very same reason, two-valued nonsovereign strategy-proof voting rules with full ballot domain do not exist.

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