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Tree-Wise Single Peaked Domains

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TREE-WISE SINGLE PEAKED DOMAINS

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Abstract. The present note provides two conditions which are jointly sufficient for a finite family of uniquely topped total preorders on a finite set to be tree-wise single peaked - even when it is not line-wise single peaked. One of the two conditions is also a necessary one.

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1. Introduction

It can be quite easily shown that -as a corollary of well-established results- the core of the majority-induced dominance relation of any (odd) profile of single peaked linear orders on a tree *includes the outcome of the sincere strategy profile, and the latter is in particular both a dominant strategy equilibrium and a strong Nash equilibrium.* Thus, single peaked domains on an *arbitrary* tree are a *coalitionally strategy-proof domain* for the majority rule: on such domains the existence of an outcome that *satisfies the Condorcet stability criterion in a remarkably robust manner* is warranted (see e.g. Danilov (1994), Vannucci (2016), building on the seminal Moulin (1980), and Demange (1982)).

Other remarkable stategy-proofness properties of probabilistic social choice rules on single peaked domains have also been pointed out (see e.g. Ehlers, Peters and Storcken (2002), Peters, Roy, Sen and Storcken (2014), concerning single peaked domains on lines, and Chatterji, Sen and Zeng (2016) for single peaked domains on trees).

However, while a rich literature concerning the specialized case of single peaked domains on *lines* is available (ranging from early work as aptly summarized in Fishburn (1973) to some recent and much more general contributions on stricly related matters such as Danilov and Koshevoy (2013) and Puppe (2014)), little is apparently known -comparatively speaking- about single peaked domains on *arbitrary* trees.

Consider for instance the most basic version of the relevant *identification problem*, namely finding sufficient conditions for an arbitrary domain of total preorders with unique maxima to be tree-wise single peaked. Its specialized version for the 'degenerate'-tree case of lines/chains has a simple and well-known solution: *each triple of elements of the ground set should include an element which is never the minimum of the triple according to some preorder of the domain* (moreover, such condition is both sufficient *and* necessary).

But what if a certain domain is clearly *not line-wise single peaked*?

Thus, we have the following version of the basic identification problem concerning tree-wise single peaked domains.

Problem Let X be a non-empty finite set, $N = \{1, ..., n\}$ and $D_X = \{\succcurlyeq_1, ..., \succcurlyeq_n\}$ a set of total preorders on X with unique maxima. Find out conditions on D_X that:

(i) are sufficient for D_X to be a *tree-wise single peaked domain* i.e. for the existence of a tree $\mathcal{T}(X)$ with X as its node-set such that each

$\succcurlyeq_i \in D_X$ is single peaked with respect to the betweenness relation of $\mathcal{T}(X)$, and

(ii): work even if D_X is *not line-wise single peaked.*

It should be emphasized that such a problem has a most significant 'practical' dimension, whenever D_X amounts to the set of admissible 'voting strategies' represented in a ballot, under any decision protocol which -like e.g., majority-based rules- is strategy-proof on single peaked domains but *not* on certain larger domains. In such a case, if D_X is single peaked then it may be plausibly regarded as reliable information about the true preferences of the relevant voters, but *not* otherwise.

Nevertheless, to the best of the author's knowledge, the foregoing Problem has never been addressed in the extant literature. The present note provides a solution to it consisting of a pair of conditions which are jointly sufficient for a domain to be tree-wise single peaked. In particular, one of those conditions is also a necessary one. The present note relies on some previous work concerning betweenness relations on trees, mainly Sholander (1952)) and Chvátal, Rautenbach, Schäfer (2011).

2. Model and result

Let X, N be finite sets, T_X the set of all binary relations $\succcurlyeq \subseteq X^2$ which are topped i.e. with a *unique maximum* $top(\succcurlyeq) \in X$. Moreover, let $\widehat{T}_X \subseteq T_X$ be the set of all *transitive* binary relations on X having a unique maximum, and T_X^* the set of all *total preorders* on X having a unique maximum. The following notation will be used: for any $\succcurlyeq_i \in T_X$, $\succ_i$and $\sim_i$denote respectively the asymmetric and symmetric components of $\succcurlyeq_i$; for any $D_X = \{\succcurlyeq_1, ..., \succcurlyeq_n\} \subseteq T_X$ and any $x \in X$, $N(D_X) = \{1, ..., n\}$, $N = N_x(D_X) = \{i \in \{1, ..., n\} : top(\succcurlyeq_i) = x\}$, and

$Top(D_X) = \{x \in X :$ there exists $\succcurlyeq_i \in D_X$ with $top(\succcurlyeq_i) = x\}$.

A ternary relation $B \subseteq X^3$ is a (*interval space*) **betweenness** on X if and only if for any $x, y, z \in X$ the following two conditions hold:

(B_0): for each $x, y, z \in X$, $(x, y, z) \in B$ whenever $y \in \{x, z\}$,

(B_1): for each $x, y, z \in X$, if $(x, y, z) \in B$ then $(z, y, x) \in B$.

A topped $\succcurlyeq_i \in T_X$ is **single peaked with respect to betweenness relation** $B \subseteq X^3$ if for each $i \in N$ and any $x, y, z \in X$, $x = top(\succcurlyeq_i)$ and $(x, y, z) \in B$ entail that $z \succ_i y$ does *not* hold. A domain $D_X \subseteq T_X$ is **single peaked** if there exists a (interval space) betweenness relation $B \subseteq X^3$ such that every $\succcurlyeq_i \in D_X$ is single peaked with respect to B.

Definition 1. *(**Tree Betweenness**) A ternary relation $B \subseteq X^3$ is a **Tree Betweenness** if and only if it satisfies the following independent conditions (see Chvátal, Rautenbach, Schäfer (2011)), Corollary 5 for a justification of the present definition):*

(B_1): for each $x, y, z \in X$, if $(x, y, z) \in B$ then $(z, y, x) \in B$;

(B_2): for each $x, y, z, w \in X$, if $(x, y, z) \in B$, $(y, z, w) \in B$ and $y \neq z$ then $(x, z, w) \in B$;

(B_3): for each $x, y, z, w \in X$, if $(x, y, z) \in B$ and $(x, z, w) \in B$ then $(y, z, w) \in B$;

(B_4): for each $x, y, z \in X$, if $B \cap \{(x, y, z), (y, z, x), (z, x, y)\} = \varnothing$ then there exists $u \in X \setminus \{x\}$ such that $(x, u, y) \in B$ and $(x, u, z) \in B$;

(B_5): for each $x, y, z \in X$, $(x, y, z) \in B$ and $(y, x, z) \in B$ if and only if $x = y$.

Remark 1. Notice that a Tree Betweenness does also satisfy B_0 hence it is a special instance of a (interval space) betweenness as defined above. To check this claim, consider any $x, y, z \in X$, $(x, y, z) \in B$ such that $y \in \{x, z\}$. Then, either $x = y$, or $z = y$. In the first case, $(x, y, z) \in B$ by B_5. In the second case, $(z, y, x) \in B$ by B_5 whence $(x, y, z) \in B$ by B_1. Moreover, consider any *partial order* $\leq$ on X. The 'canonical' *order-betweenness* relation $B^{\leq} \subseteq X^3$ is defined in the obvious way, namely

$B^{\leq} := \{(x, y, z) \in X^3\colon\ x \leq y \leq z$ or $z \leq y \leq x$, or $y \in \{x, z\}\}$.

It is quite easy -and left to the reader- to check that if $\leq$ is a *linear order* then $B^{\leq}$ (a *Line Betweenness,* by definition) does satisfy properties $B_1 - B_5$ i.e. it is indeed a special instance of a Tree Betweenness.

Definition 2. *(**Tree-wise Single Peaked domains**) A finite domain $D_X = \{\succcurlyeq_1, ..., \succcurlyeq_n\} \subseteq T_X$ is is **Tree-wise Single Peaked (TSP)** if there exists a Tree Betweenness $B \subseteq X^3$ such every $\succcurlyeq_i \in D_X$ is single peaked with respect to B.*

Let us now consider the following two conditions on a domain $D_X \subseteq T_X$:

Compromise Availability for Triplets *(CAT):* for any $x, y, z \in Top(D_X)$, if there exist $i^x, i^y, i^z \in N(D_X)$ such that

$x = \min(\succcurlyeq_{i^x}) \{x, y, z\}$, $y = \min(\succcurlyeq_{i^y}) \{x, y, z\}$ and $z = \min(\succcurlyeq_{i^z}) \{x, y, z\}$, then for some $u \neq x$ both $u \neq \min(\succcurlyeq_i) \{x, u, y\}$ and $u \neq \min(\succcurlyeq_i) \{x, u, z\}$ hold for each $i \in N(D_X)$.

Consistency of Local Unanimity on Minima *(CLUM):* for any $x, y, z, w \in Top(D_X)$,

if $y \neq \min(\succcurlyeq_i)\{x, y, z\}$ and $z \neq \min(\succcurlyeq_i)\{x, z, w\}$ for each $i \in N(D_X)$, then $z \neq \min(\succcurlyeq_i)\{y, z, w\}$ for each $i \in N(D_X)$.

Example. Take $X = \{x, y, z, u\}$ and consider domain $D_X = \{\succcurlyeq_1, \succcurlyeq_2, \succcurlyeq_3\} \subseteq T_X^*$ such that:

$x \succ_1 u \succ_1 y \succ_1 z$
$y \succ_2 z \succ_2 u \succ_2 x$
$z \succ_3 u \succ_3 x \succ_3 y$.

It is easily checked that $(\succcurlyeq_1, \succcurlyeq_2, \succcurlyeq_3)$ satisfies CAT, by construction. Moreover, it also trivially satisfies CLUM since $Top(\{\succcurlyeq_1, \succcurlyeq_2, \succcurlyeq_3\})k = \{x, y, z\}$.

However, it can be easily shown that -due to its 'disconnected' triplet x, y, z- there is *no linear order* $\leq$ on X such that $\{\succcurlyeq_1, \succcurlyeq_2, \succcurlyeq_3\}$ is *line-wise single peaked* with respect to order-betweenness $B^{\leq}$. Thus CAT and CLUM may indeed jointly hold for domains which are *not line-wise single peaked.*

Theorem 1. *(i) Let domain $D_X \subseteq T_X^*$ satisfy CAT and CLUM. Then, D_X is a TSP domain.*

(ii) Moreover, if D_X is a TSP domain then it satisfies CAT.

Proof. Part (i):

Let us define a ternary relation $B(D_X) \subseteq X^3$ as follows: for any $x, y, z \in X$,

$(x, y, z) \in B(D_X)$ if and only if $\{x, z\} \subseteq Top(D_X)$ and either
(α) $y \in \{x, z\}$ or
(β) $y \neq \min(\succcurlyeq_i)\{x, y, z\}$ for all $i \in N(D_X)$.

To begin with, notice that D_X is indeed a *single peaked domain with respect to* $B(\succcurlyeq^N)$.

Indeed, let $(x, y, z) \in B(D_X)$. Then, $\{x, z\} \subseteq Top(D_X)$ and either $y \in \{x, z\}$ or $y \neq \min(\succcurlyeq_i)\{x, y, z\}$ for all $i \in N(D_X)$. Suppose first that $y \in \{x, z\}$ and $i \in N_x(D_X)$ i.e. $top(\succcurlyeq_i) = x$: then, $y \succ_i z$ if $y = x \neq z$ and $y \sim_i z$ otherwise. Suppose now that $y \neq \min(\succcurlyeq_i)\{x, y, z\}$ i.e. either $y \succcurlyeq_i x$ or $y \succcurlyeq_i z$ for all $i \in N(D_X)$ (and *not* $x = y = z$). Then, consider any $i \in N_x(D_X)$: either $y = x$, whence $y \succ_i z$ or $y \neq x$ whence $y \succcurlyeq_i z$, and single peakedness of $\succcurlyeq^N$ holds.

We claim that $B(D_X)$ does indeed satisfy properties B_i, $i = 1, ..., 5$.

B_1: Immediate. Indeed, suppose that $(x, y, z) \in B(D_X)$: then $\{x, z\} \subseteq Top(D_X)$ and either $y \in \{x, z\}$ or $y \neq \min(\succcurlyeq_i)\{x, y, z\}$ for all $i \in N$. In both cases $(z,y,x) \in B(D_X)$ by the very definition of $B(D_X)$.

B_2: Suppose that $x,y,z,w \in X$, $y \neq z$, $(x,y,z) \in B(D_X)$ and $(y,z,w) \in B(D_X)$. In view of $B(D_X)$'s definition we have to distinguish four cases corresponding to the possible combinations of clauses, namely:

(a) $y \in \{x,z\}$ and $z \in \{y,w\}$. Then, since $y \neq z$, $y = x$ and either $z = y$ or $z = w$. Hence either case $z \in \{x,w\}$ hence $(x,z,w) \in B(D_X)$ by definition of $B(D_X)$, clause (α).

(b) $y \in \{x,z\}$ and $z \neq \min(\succcurlyeq_i)\{y,z,w\}$ for all $i \in N(D_X)$. Since $y \neq z$ it follows that $y = x$. Thus $z \neq \min(\succcurlyeq_i)\{x,z,w\}$ for all $i \in N(D_X)$, whence $(x,z,w) \in B(D_X)$ by definition of $B(D_X)$, clause (β).

(c) $y \neq \min(\succcurlyeq_i)\{x,y,z\}$ for all $i \in N(D_X)$ and $z \in \{y,w\}$. Again, it must be the case that $z = w$, since $y \neq z$. Thus, $z \in \{x,w\}$ hence $(x,z,w) \in B(D_X)$ by definition of $B(D_X)$, clause (α).

(d) $y \neq \min(\succcurlyeq_i)\{x,y,z\}$ and $z \neq \min(\succcurlyeq_i)\{y,z,w\}$ for all $i \in N(D_X)$. Suppose then that there exists $j \in N(D_X)$ such that $z = \min(\succcurlyeq_j)\{x,z,w\}$ i.e. without loss of generality both $x \succ_j z$ and $w \succ_j z$ (indeed, if $z \in \{x,w\}$ then again $(x,z,w) \in B(D_X)$ by definition of $B(D_X)$, clause (α)). But $w \succ_j z$ and $z \neq \min(\succcurlyeq_i)\{y,z,w\}$ for all $i \in N(D_X)$ with $y \neq z$ entail $z \succ_j y$. However, $y \neq \min(\succcurlyeq_j)\{x,y,z\}$ by hypothesis, hence $y \succ_j x$. Since $x \succ_j z$ by hypothesis, it follows -by transitivity- that $y \succ_j z$, a contradiction. Therefore, again, $z \neq \min(\succcurlyeq_i)\{x,z,w\}$ for all $i \in N(D_X)$, whence $(x,z,w) \in B(D_X)$ by definition of $B(D_X)$, clause (β). As a consequence B_2 holds.

B_3: Suppose that $(x,y,z) \in B(D_X)$ and $(x,z,w) \in B(D_X)$. We claim that $(y,z,w) \in B(D_X)$ as well. To check this, we distinguish again the four possible cases, namely:

(a) $y \in \{x,z\}$ and $z \in \{x,w\}$. Then, if $y = x$ then both $z = x$ and $z = w$ entail $z \in \{y,w\}$ whence $(y,z,w) \in B(D_X)$ by clause (α). Otherwise, $y = z$ hence again $z \in \{y,w\}$ and $(y,z,w) \in B(D_X)$ by clause (α).

(b) $y \in \{x,z\}$ and $z \neq \min(\succcurlyeq_i)\{x,z,w\}$ for all $i \in N(D_X)$. If $y = x$ then $z \neq \min(\succcurlyeq_i)\{y,z,w\}$ for all $i \in N(D_X)$ hence $(y,z,w) \in B(\succcurlyeq^N)$ by clause (β). If $y = z$ then $z \in \{y,w\}$ and $(y,z,w) \in B(D_X)$ by clause (α).

(c) $y \neq \min(\succcurlyeq_i)\{x,y,z\}$ for all $i \in N(D_X)$ and $z \in \{x,w\}$. If $z = x$ then $y \succ_i x = z$ for each $i \in N(D_X)$: a contradiction, by hypothesis, $\{x,z\} \subseteq Top(D_X)$. Therefore, $z = w$ whence $z \in \{y,w\}$ and $(y,z,w) \in B(D_X)$ by clause (α) again.

(d) $y \neq \min(\succcurlyeq_i)\{x,y,z\}$ and $z \neq \min(\succcurlyeq_i)\{x,z,w\}$ for all $i \in N(D_X)$. Then, by property CLUM of domain D_X, $z \neq \min(\succcurlyeq_i)\{y,z,w\}$

for each $i \in N(D_X)$ hence $(y, z, w) \in B(D_X)$ by clause (β). It follows that B_3 holds.

B_4: Let $x, y, z \in X$ be such that
$B(D_X) \cap \{(x, y, z), (y, z, x), (z, x, y)\} = \varnothing$. Then, by definition of $B(D_X)$, $x \neq y \neq z \neq x$ and there exist $i^x, i^y, i^z \in N(D_X)$ such that $x = \min(\succcurlyeq_{i^x}) \{x, y, z\}$, $y = \min(\succcurlyeq_{i^y}) \{x, y, z\}$ and $z = \min(\succcurlyeq_{i^z}) \{x, z, w\}$. Hence, by property CAT of domain D_X, there exists $u \neq x$ such that both $u \neq \min(\succcurlyeq_i) \{x, u, y\}$ and $u \neq \min(\succcurlyeq_i) \{x, u, z\}$ hold for each $i \in N(D_X)$. But then, both $(x, u, y) \in B(D_X)$ and $(x, u, z) \in B(D_X)$ by clause (β), and B_4 is also satisfied.

B_5: $\Longrightarrow$ Suppose that both $(x, y, z) \in B(D_X)$ and $(y, x, z) \in B(D_X)$. Again, by definition of $B(D_X)$, $\{x, y, z\} \subseteq Top(D_X)$ and there are of course four distinct cases to consider, namely:

(a) $y \in \{x, z\}$ and $x \in \{y, z\}$. If $x = y$ there is nothing to prove, and if $y = z$ and $x = z$ then of course $x = y$.

(b) $y \in \{x, z\}$ and $x \neq \min(\succcurlyeq_i) \{x, y, z\}$ for all $i \in N(D_X)$. Suppose $y = z$. Then, $x \succ_i y = z$ for all $i \in N(D_X)$, a contradiction since $\{y, z\} \subseteq Top(D_X)$. Thus, $y = x$ as required.

(c) $y \neq \min(\succcurlyeq_i) \{x, y, z\}$ for all $i \in N(D_X)$ and $x \in \{y, z\}$. Suppose $x = z$. Then, $y \succ_i x = z$ for all $i \in N(D_X)$, a contradiction since $\{x, z\} \subseteq Top(D_X)$. Hence $x = y$, again.

(d) $y \neq \min(\succcurlyeq_i) \{x, y, z\}$ and $x \neq \min(\succcurlyeq_i) \{x, y, z\}$ for all $i \in N(D_X)$. In this case, $z = \min(\succcurlyeq_i) \{x, z, w\}$ for all $i \in N(D_X)$, a contradiction since by hypothesis $z \in Top(D_X)$. It follows that $x = y$ as required.

$\Longleftarrow$ Suppose $x = y$. Then, of course, $x \in \{y, z\}$ and $y \in \{x, z\}$ whence both $(x, y, z) \in B(D_X)$ and $(y, x, z) \in B(D_X)$ by definition of $B(D_X)$, clause (α).

Therefore, $B(D_X)$ satisfies property $B_1 - B_5$, hence it is indeed a *tree betweenness*. It follows that D_X is a TSP profile as required.

Part (ii):

Suppose D_X i.e. there exists a tree betweenness B such that any $\succcurlyeq_i \in D_X$ is single peaked with respect to B. Now, consider any $x, y, z \in X$. Since B satisfies property B_4, it must be the case that at least one of the following conditions holds true: (α) $(x, y, z) \in B$; (β) $(x, y, z) \notin B$ and $(y, z, x) \in B$; (γ) $(x, y, z) \notin B$ and $(z, x, y) \in B$;

(δ) $B \cap \{(x, y, z), (y, z, x), (z, x, y)\} = \varnothing$ and there exists $u \in X$, $u \neq x$ such that both $(x, u, y) \in B$ and $(x, u, z) \in B$. Next, suppose that D_X violates CAT. Then, there exist a triplet $x', y', z' \in X$ such that $\{x', y', z'\} \subseteq Top(D_X)$, and $i^{x'}, i^{y'}, i^{z'} \in N(D_X)$ such that $x' = \min(\succcurlyeq_{i^{x'}}) \{x', y', z'\}$, $y' = \min(\succcurlyeq_{i^{y'}}) \{x', y', z'\}$ and $z' = \min(\succcurlyeq_{i^{z'}}) \{x', y', z'\}$, and for every $u \neq x'$ both $u = \min(\succcurlyeq_i) \{x', u', y\}$ and

$u = \min(\succcurlyeq_j)\{x', u, z'\}$ hold for some $i, j \in N(D_X)$. As observed above, (x', y', z') must satisfy at least one of conditions $(\alpha), (\beta), (\gamma), (\delta)$. But, as it is easily checked, if any one of such conditions holds for triplet (x', y', z') it follows that D_X is *not* TSP, a contradiction. □

Remark 2. It should be noticed that CAT and CLUM are mutually independent. To check the validity of this statement, consider the following two profiles of topped total preorders on X:

(i) $(\succcurlyeq_1, \succcurlyeq_2, \succcurlyeq_3)$ where $\succcurlyeq_i, i = 1, 2, 3$ are such that, for all $a, b \in X \setminus \{x, y, z\}$:

$x \succ_1 y \succ_1 z \succ_1 a \sim_1 b$,
$y \succ_2 z \succ_2 x \succ_2 a \sim_2 b$,
$z \succ_3 x \succ_3 y \succ_3 a \sim_3 b$.

Clearly, by construction, $\{\succcurlyeq_1, \succcurlyeq_2, \succcurlyeq_3\}$ satisfies CLUM but violates CAT.

(ii) $(\succcurlyeq_1, \succcurlyeq_2)$ where $\succcurlyeq_1 = \succcurlyeq_2 = \succcurlyeq$ and $\succcurlyeq$ such that $w \succ y \succ z \succ x \succ a \sim b$ for all $a, b \in X \setminus \{x, y, z, w\}$. It is easily checked that $(\succcurlyeq_1, \succcurlyeq_2)$ satisfies CAT but violates CLUM.

It would be interesting to find necessary and sufficient conditions for $D_X \subseteq T_X^*$ to be a TSP domain, and sufficient and/or necessary conditions for D_X to be TSP with $D_X \subseteq \widehat{T}_X$ and $D_X \subseteq T_X$ respectively. Those issues however are beyond the scope of the present note, and are left as a topic for further research.

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