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Aggregation of coherent experts opinion: a tractable extreme-outcomes consistent rule

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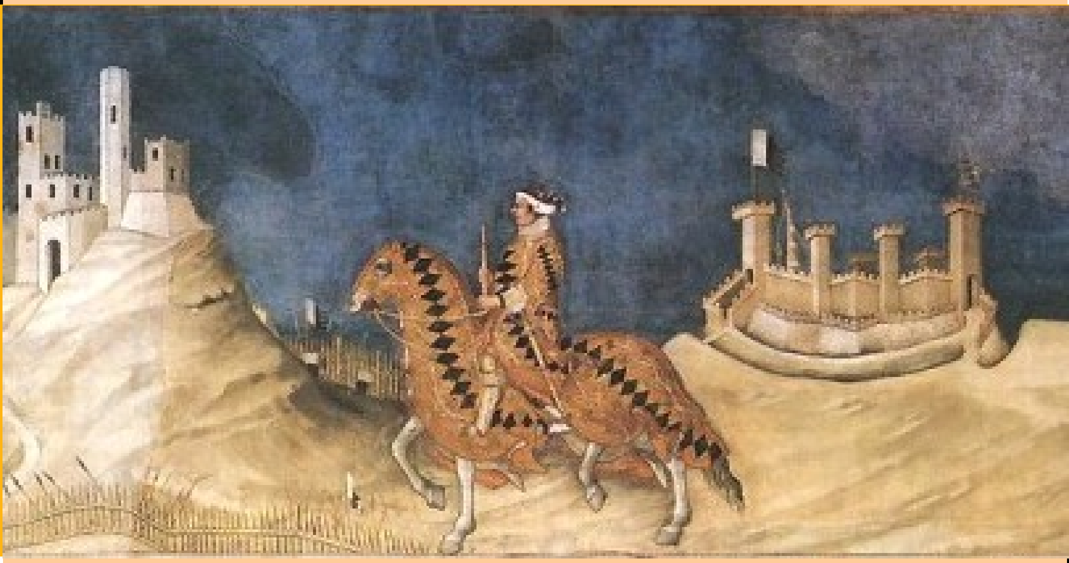

Abstract - The paper defines a consensus distribution with respect to experts' opinions by a multiple quantile utility model. The paper points out that the Steiner Point is the representative consensus probability. The new rule of experts' opinions aggregation, that can be evaluated by the Shapley value in a simple way, is prudential and coherent.

Keywords: Ambiguity, Aggregation, Steiner Point, Multiple Priors, Quantiles.

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1 Introduction

In the last decades it has been observed a new interest for aggregation of conflicting and not necessarily independent opinions of experts and scientists when the decision maker (DM) faces ambiguity. Examples of ambiguous events in which estimations are derived from empirical frequencies that induce not fully reliable and unique assessment among experts are: the severity of the global warming following a doubling of atmospheric CO_2 concentration, with respect to preindustrial levels, the relationship between biodiversity loss-environmental services change, and transmission of re-emerging infectious diseases, the environmental safety and direct effects on human health of genetically modified organisms, the morbidity and mortality of a pandemic flu in human beings.

Because of difficulties and failure of the frequency theory respect incomplete, sparse or unavailable data, the Bayesian theory emerged as the normative theory able to solve the problem of opinions aggregation. In the Bayesian context, the concept of opinion, that encompasses different notions such as: prior, vague prior, sequence of odds ratios, finitely additive measures finds a natural encode in a subjective probability distribution.[1]

Nevertheless, in Bayesian axiomatic approach to consensus distribution there is no room for ambiguity, but ambiguity attitude, that emerges when individuals face vague and incomplete statistical data, influences perception of risky events and induces human beings to elicit probabilities and apply decision rules that violate the standard rationality paradigm. Recently, a stream of processes involving both modeling and behavioral aspects has been proposed to calibrate the aggregation-combination-composition of experts' opinion through the DM's ambiguity attitude. These methods differ from Bayesian pooling operators used to form a single consensus distribution and can be included in three main classes: (i) pooling methods based on Dempster's rule of combination or theory of evidence (Stephanou and Lu 1988; Bi *et al.* 2007; HaDuong 2008; Denoeux 2008); (ii) combination rules based on possibility distributions and fuzzy measures (Sandri *et al.* 1993; Dubois and Prade 1994 ; Yu 1997); (iii) methods of aggregation based on multiple priors or capacity (Gajdos and Vergnaud 2012, Cres *et al.* 2011).[2]

We consider aggregation scheme of opinions expressed through different probability distributions and a DM that adopts a multiple priors decision model.

[1] Whether opinions are expressed as probability measures, densities, mass functions, odds then subjective probability distributions are used to form a consensus distribution through: linear opinion pools (Stone 1961), linear opinion pools, that only satisfies the *marginalization property,* (DeGroot and Montera 1991), logarithmic opinion pools, that satisfies the *external Baysianity* (Winkler 1968), generalized logarithmic opinion pools (Genest *et al.* 1986), that is using mathematical aggregation models.

[2] Gajdos and Vergnaud (2012) characterize preferences that exhibit independently aversion towards imprecision and conflict, axiomatically. They aggregate different evaluations through a multiple weights model where "a decision maker will satisfy the conflict aversion hypothesis whenever her degree of conflict aversion is higher than her degree of imprecision aversion". Cres *at al.* (2011) focus on the maxmin expected utility model so that "the decision maker's valuation of an act is the minimal weighted valuation, over all weight vectors of the experts' valuations" (a weight pessimistic scenario).

Following the standard literature, the set of probability distributions or lotteries of all experts can be considered reflecting the DM's assessment of the reliability of available information about the underlying uncertainty, that is, her perception of ambiguity, and the optimal aggregation rule incorporates the DM's attitude about scanty and vague information. Facing the set of all probability distributions attached by experts to possible events, the DM considers the mean value of their common probability set, indeed the mean value of their probabilities intersection. Such a mean value is the *Steiner point* of the convex capacity that emerges from experts opinion aggregation. *Section 2* introduces an aggregation rule based on a quantile utility model and a DM that is supposed to be pessimistic with respect to extreme negative outcomes (catastrophic losses), ambiguity neutral in an interval of more reliable outcomes (familiar results) and optimistic with respect to extreme positive outcomes (windfall gains). In *Section 3* we translate in terms of attitude toward lower tails, upper tails and intermediate quantiles and define a less conservative criterion for eliciting a single consensus distribution. Moreover we show that the suggested aggregation rule preserves stochastic dominance. Concluding remarks follows.

2 Opinions Aggregation in a Multiple Quantile Utility Model

Talking about the use of proper pooling methods to discover not only opinion of people's, whom the DM regards as experts; but also to judge how well informed they are, Savage observes that "risks characterized by tiny probabilities may be difficult to have a reliable experts' assessment, that experts' opinions might be divergent, and, what is more relevant, you might discover with expert is optimistic or pessimistic in some respect and therefore temper his judgements. Should he suspect you of this, however, you and he may well be on the escalator to perdition" (Savage 1971, 796). Moreover there exist some problems to the full implementation of consensus rules in an axiomatic Bayesian approach to the expert priors and the DM's prior updating by a likelihood function, indeed the arbitrariness of the pooling weights, the use of invariant combination rules, the dependence between the DM's information and the experts' information, dependence among experts' probability distributions (e.g. stochastic dependence), and calibration of experts' opinion.[3]

We introduce an approach to form a consensus distribution that adopts a quantile-function (Basili and Chateauneuf 2011). We give a representation in a setting with a DM that has multiple priors, indeed the set of all probability distributions of each expert on possible events, none of which is considered fully reliable. We put in evidence a new formalization of the aggregation rule

[3]In empirical studies Clemen and Winkler (1986) and Figlewski and Urich (1983) found that correlations among expert forecasts can be above 0.80.

that rests on the idea that the decision-maker has a set of outcomes called ordinary, because they are considered more reliable (*familiar* or closer to her experienced life), and two fat tails in which are included more ambiguous extreme (*unfamiliar*) events. Differently from other possible formulations, our new formulation appears to be less conservative and extreme: the DM is supposed to be pessimistic with respect to purely catastrophic losses, ambiguity neutral with respect to ordinary outcomes and optimistic with respect to purely windfall gains.

2.1 Framework

We consider m experts $j = 1, ..., m$ aiming at valuing the possible probability distributions P governing an uncertain situation $S = \{s_1, ..., s_i, ...s_n\}$, where one state $s \in S$ will occur and only one, but where it is assumed that there is a unique unknown probability distribution P_0 governing that situation. Formally, S is the finite set of states of the world and 2^S is the set of all subset of S. Any given expert i, will be asked to give *lower* and *upper bounds* for the probability $p_0^i = P_0(\{i\})$. Therefore the set of possible probabilities P_j considered by expert j will be $P_j = \left\{P = (p_1, ..., p_i, ..., p_m),\ a_i^j \leq p_i \leq b_i^j,\ i = 1, ..., n\right\}$.

We indeed assume $0 \leq a_i^j \leq b_i^j \leq 1$.

It is straightforward that $P_j \neq \varnothing$ if and only if $\sum_i a_i^j \leq 1 \leq \sum_i b_i^j$ [1]

As proved in Chateauneuf and Cornet (2012), it turns that as soon as $P_j \neq \varnothing$, then P_j is the the *core* $C(\upsilon_j)$ of a convex capacity υ_j which turns out to be defined easily, namely:

$$\forall A \in 2^S,\ \upsilon_j(A) = Max\left(\sum_{i \in A} a_i^j,\ 1 - \sum_{i \notin A} b_i^j\right).$$

Due to *fiability* of experts, even if they do not know P_0 they should envision a set $\mathcal{P}_j$, such that $P_0 \in \mathcal{P}_j$. Therefore one should expect that $\bigcap_j \mathcal{P}_j \neq \varnothing$.

So a first test to validate the quality (competence and reliability) of the panel of experts should be to check that $\bigcap_j \mathcal{P}_j \neq \varnothing$.

From [1], it is immediate that $\bigcap_j \mathcal{P}_j \neq \varnothing$ is equivalent to $\sum_i a_i \leq 1 \leq \sum_i b_i$, where: $a_i = \underset{j}{Max}\, a_i^j$, $b_i = \underset{j}{\min}\, b_i^j$ and indeed $a_i \leq b_i\ \forall i$.

Once these conditions have been checked or, if $\bigcap_j \mathcal{P}_j = \varnothing$, the experts have been asked to revise their opinion by enlarging their considered initial $\mathcal{P}_j$, in order to satisfy the consistency requirement $\bigcap_j \mathcal{P}_j \neq \varnothing$, one could summarize the *consensus opinion* $P = \bigcap_j \mathcal{P}_j$ through a convex capapcity[4] υ with the known formula $\upsilon(A) = Max\left(\sum_{i \in A} a_i,\ 1 - \sum_{i \notin A} b_i\right)$.

[4]A capacity υ is convex if $\upsilon(A \cup B) + \upsilon(A \cap B) \geq \upsilon(A) + \upsilon(B), \forall A, B \in 2^S$

In other words this convex capacity υ will be now considered as the aggregation of the multiple prior opinions.

2.2 Multiple Quantile Utility Model

We now suggest to use the multiple quantile utility model (as considered in Basili and Chateauneuf 2011) with respect to the previous convex capacity. In this way we show that the Steiner point $\Pi^{\vartheta} \in C(\upsilon)$ can be considered as the representative probability of the consensus experts' opinions.

As a matter of fact the Steiner point[5] is defined as the *center of* $C(\upsilon)$, so as a meaningful probability summarizing the consensus experts' opinions. Moreover it turns out that for convex capacities the Steiner point is nothing else that the famous *Shapley value*, the computation of which is very easy.[6]

Let us recall (i.e. Owen 1968) that Shapley's Π^{Sh} (i.e. $\Pi^{Sh} = \Pi^{\vartheta}$) is defined by:

$$\forall i \in [1,n] \;\; \Pi_i^{\vartheta} = \sum_{\{i\} \subset A \subset S} \tfrac{(|A|-1)!(n-|A|)!}{n!} \left[\upsilon(A) - \upsilon(A \backslash \{i\})\right]$$

Example 1 *Computation of the Shapley value for the following 'probability-interval' capacity* υ

$$\begin{array}{lcccc} S & = & s_1 & s_2 & s_3 \\ b_i & = & \frac{6}{12} & \frac{5}{12} & \frac{7}{12} \\ a_i & = & \frac{2}{12} & \frac{3}{12} & \frac{4}{12} \end{array} \qquad \text{therefore } \upsilon \text{ is given by}$$

A	$\{s_1\}$	$\{s_2\}$	$\{s_3\}$	$\{s_1, s_2\}$	$\{s_1 s_3\}$	$\{s_2 s_3\}$	S
$\upsilon(A)$	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{4}{12}$	$\frac{5}{12}$	$\frac{7}{12}$	$\frac{7}{12}$	$\frac{12}{12}$

therefore

$$\Pi_1^{\vartheta} = \frac{1}{6}\left\{2.\frac{2}{12} + \frac{1.(5-3+7-4)}{12} + \frac{2.(12-7)}{12}\right\} = \frac{19}{72}$$

$$\Pi_2^{\vartheta} = \frac{1}{6}\left\{2.\frac{3}{12} + \frac{1.(5-2+7-3)}{12} + \frac{2.(12-5)}{12}\right\} = \frac{27}{72}$$

$$\Pi_3^{\vartheta} = \frac{1}{6}\left\{2.\frac{4}{12} + \frac{1.(7-2+7-4)}{12} + \frac{2.(12-7)}{12}\right\} = \frac{26}{72}$$

hence $\Pi_i^{\vartheta} = (\frac{19}{72}, \frac{27}{72}, \frac{26}{72})$.

3 A coherent prudential aggregation rule of experts opinions

According to assumption, the DM's ambiguity attitude is modeled through a convex capacity υ. For an act $X : S \to \mathbb{R}$ and $(\alpha, \beta) \in [0,1]^2, \alpha \leq \beta$, such that $[\alpha, \beta]$ determines the interval of cumulative probability between which outcomes can be considered as ordinary, the DM values outcomes in $[\alpha, \beta]$ in an ambiguity neutral way by Π^{ϑ}. With respect to ambiguity attitude on extreme outcomes,

[5] Details are in Schneider 1971 and 1993.
[6] Details are in Gajdos et al 2008.

we propose a prudential aggregation rule: namely one that assumes pessimism on lower tail $[0,\alpha]$ and optimism on the upper tail $[\beta,1]$.

For any act, it is defined the common pseudo-inverse F_X^{-1}, indeed the quantile-function.[7]

As a result, the DM chooses $\alpha,\beta \in [0,1]$, where $\alpha \leq \beta$, and computes the value of $X \in \mathbb{R}^S$ through $I(X) = I_1(X) + I_2(X) + I_3(X)$, where, defined $\overline{v}$ the conjugate of v[8]: $I_1(X) = \int_0^\alpha F_X^{v^{-1}}(p)dp$; $I_2(X) = \int_\alpha^\beta F_X^{\Pi^{\vartheta^{-1}}}(p)dp$ and $I_3(X) = \int_\beta^1 F_X^{\overline{v}^{-1}}(p)dp$.

Once the probability distribution $\Pi^{Sh} = \Pi^\vartheta \in core(v)$ has been selected, it is possible to define the DM pessimism with respect to outcomes in the lower tail and optimism with respect to outcomes in the upper tail. In fact, pessimism and optimism are defined with respect to the probability distribution that expresses ambiguity neutrality, i.e. Π^ϑ.

Definition 1 *The DM is pessimistic with respect to the lower tail if she overestimates losses and underestimates gains in this tail with respect to the probability $\Pi^\vartheta \in C(v)$, i.e., if $I_1(X) \leq \int_0^\alpha F_X^{\Pi^{\vartheta^{-1}}}(p)dp$.*

Definition 2 *The DM is optimistic with respect to the upper tail if she underestimates losses and overestimates gains in this tail with respect to the probability $\Pi^\vartheta \in C(v)$, i.e., if $I_3(X_i) \geq \int_\beta^1 F_X^{\Pi^{\vartheta^{-1}}}(p)dp$.*

Proposition 1 *Under the prudential rule, the DM is pessimistic with respect to the lower tail and optimistic with respect to the upper tail.*[9]

Interesting enough, this rule could be particularly useful to describe potential global temperatures rise and climate sensitivity after recent evidence that five year mean global temperature has been flated for a decade. In case this rule appears as prudential or cautious in the sense that estimations for high temperatutes would be overvaluated while estimations for low temperatutes would give lower values than the consensus one's.

Stochastic dominance provides a powerful method for act analysis since it does not require assumptions concerning the shape of the probability function or utility function and utilizes every point in the set of probability distributions.

Furthermore it is known that under risk one of the rare rules upon which decision theorists agree, is the respect of first order stochastic dominance. So we introducethe following definition.

Definition 3 *A rule $I : \mathbb{R}^S \to \mathbb{R}$ is said to be coherent with the probabilistic information ϑ if given $X, Y \in \mathcal{F}$, X first order stochastically dominates Y for any probability $P \in C(v)$ i.e. if $P(X \geq t) \geq P(Y \geq t) \quad \forall t \in \mathbb{R}$ and $\forall P \in C(v)$ implies $I(X) \geq I(Y)$.*

[7] Details are in Basili and Chateauneuf 2011.
[8] The dual or conjugate capacity $\overline{v}$ is defined by $\overline{v}(A) = 1 - v(A^C)\ \forall A \in 2^S$.
[9] *Proposition 1* is a direct consequence of *Proposition 3* in Basili and Chateauneuf 2011.

Proposition 2 *The prudential rule is coherent.*

Proof. Let $X, Y \in \mathcal{F}$ such that X first order stochastically dominates Y for any probability $P \in C(v)$. Let us first show that $I_1(X) \geq I_1(Y)$. Recall that for $p \in [0,1]$, $F_X^{\vartheta^{-1}}(p) = \inf \left\{ t \in \bar{\mathbb{R}}, 1 - \vartheta(X > t) \geq p \right\}$ and $F_X^{\vartheta}(t) = \bar{\vartheta}(X \leq t) = 1 - \vartheta(X > t)$ $\forall t \in \mathbb{R}$. Take $\underline{P} \in C(v)$ then $\underline{P}(X > t) \geq \underline{P}(Y > t)$ so $\min\limits_{P \in C(v)} P(X > t) \geq \min\limits_{P \in C(v)} P(Y > t)$. Since ϑ is convex hence exact this implies $\vartheta(X > t) \geq \vartheta(Y > t)$ hence $F_X^{\vartheta}(t) \leq F_Y^{\vartheta}(t)$ and therefore $F_X^{\vartheta^{-1}}(p) \geq F_Y^{\vartheta^{-1}}(p)$, $\forall p \in [0, \alpha]$ which gives $I_1(X) \geq I_1(Y)$.

Similar proofs apply for I_2 and I_3 which completes the proof ■

4 Concluding remarks

We developed a new approach to form a consensus distribution by considering the composite inverse cumulative function. We put in evidence that a rational DM would aggregate experts' probability distributions in a functional that combines her different attitude with respect to likely and extreme events. The new functional allows to represent ambiguity attitude about experts competence on uncertain events under scrutiny. The functional form overcomes misevaluation induced by cognitive insensitivity to small probability outcomes. Finally, our prudential rule preserves stochastic dominance.

5 References

Basili M., and A. Chateauneuf (2011), Extreme events and entropy: a multiple quantile utility model, *International Journal of Approximate Reasoning*, 52, 1095–1102.

Bi Y., Bell D., Wang H., Guo G., and J. Guan (2007), Combining multiple classifier using Dempster's rule for text categorization, *Applied Artificial Intelligence*, 21, 211-239.

Chateauneuf A. and B. Cornet (2012), Computing the cost spread of a financial structure with bid/ask, Working Paper Universite Paris 1 and University of Kansas.

Clemen R., and R. Winkler (1986), Combining Economic Forecasts, *Journal of Business and Economic Statistics*, 4, 39-46.

Cooke R. (1991), Experts in Uncertainty: Opinion and Subjective Probability in Science, Oxford University Press, Oxford.

Crès H., Gilboa I., and N. Vieille (2011), Aggregation of Multiple Prior Opinions, *Journal of Economic Theory*, 146, 2563-2582.

Darroch J., and D. Ratcliff (1972), Generalized Iterative Scaling for Log-Linear Models, *Annals of Mathematical Statistics*, 43, 1470-1480.

DeGroot M., and J. Montera (1991), Optimal Linear Opinion Pools, *Management Science*, 37, 546-558.

Denoeux T. (2008), Conjunctive and disjunctive combination of belief functions induced by non distinct bodies of evidence. *Artificial Intelligence*, 172, 234-264.

Dubois D., and H. Prade (1994), Possibility theory and data fusion in poorly informed environment. *Control Engineering Practice*, 2, 812-823.

Figlewski S., and T. Urich (1983), Optimal aggregation of money supply forecasts: accuracy, profitability and market efficiency, T*he Journal of Finance,* 28, 695-710.

Gajdos T., T. Hayashi, J.-M. Tallon, and J.-C. Vergnaud (2008), Attitude toward imprecise information, *Journal of Economic Theory*, 140, 27–65.

Gajdos T., and J. Vergnaud (2012), Decisions with conflicting and imprecise information, *Social Choice and Welfare,* DOI 10.1007/s00355-012-0691-1

Genest C., McConway K, and M. Schervish (1986), Characterization of Externally Bayesian Pooling Operators, *Annals of Statistics*, 14, 487-501.

Genest C., and B. Remillard (2006), Comments on T. Mikosh's paper "Copulas: Tales and facts", *Extremes*, 9, 27-36.

Genest C., and J. Zidek (1986), Combining Probability Distributions: A Critique and an Annotated Bibliography, *Statistical Science*, 1, 114-148.

Gilboa I. (1988). A combination of expected utility and maxmin decision criteria, *Journal of Mathematical Psychology*, 32, 405-420.

Ha-Duong M. (2008), Hierarchical fusion of expert opinion in the Transferable Belief Model, application to climate sensitivity. *International Journal of Approximate Reasoning*, 49, 555-574.

Owen G. (1968), Game Theory, Saunders.

Sandri S., Dubois D., and H. Kalfsbeek (1995), Corrections to Elicitation, Assessment, and Pooling of Expert Judgments Using Possibility Theory, *Fuzzy Systems, IEEE Transactions on,* 3, Identifier 10.1109/TFUZZ.1995.4819

Savage L. (1971), Elicitation of Personal Probabilities and Expectations, *Journal of the American Statistical Association*, 66, 783-801.

R. Schneider (1971), On Steiner points of convex bodies, *Israel Journal of Mathematics*, 2, 241-249.

R. Schneider (1993) Convex Bodies: the Brunn-Minkowski Theory, Cambridge, Cambridge University Press.

Stephanou H., and S. Lu (1988), Measuring consensus effectiveness by a generalized entropy criterion, *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, 10, 544–554.

Stone M. (1961), The Opinion Pool, *Annals of Mathematical Statistics*, 32, 1339-1342.

Winkler R. (1968) The Consensus of Subjective Probability Distributions, *Management Science*, 15, 61-75 .

Winkler R. (1986). Expert Resolution, *Management Science*, 32, 298-303.

Yu D. (1997), Expert opinion elicitation process using a fuzzy probability, *Journal of the Korean Nuclear Society,* 29, 25-34.