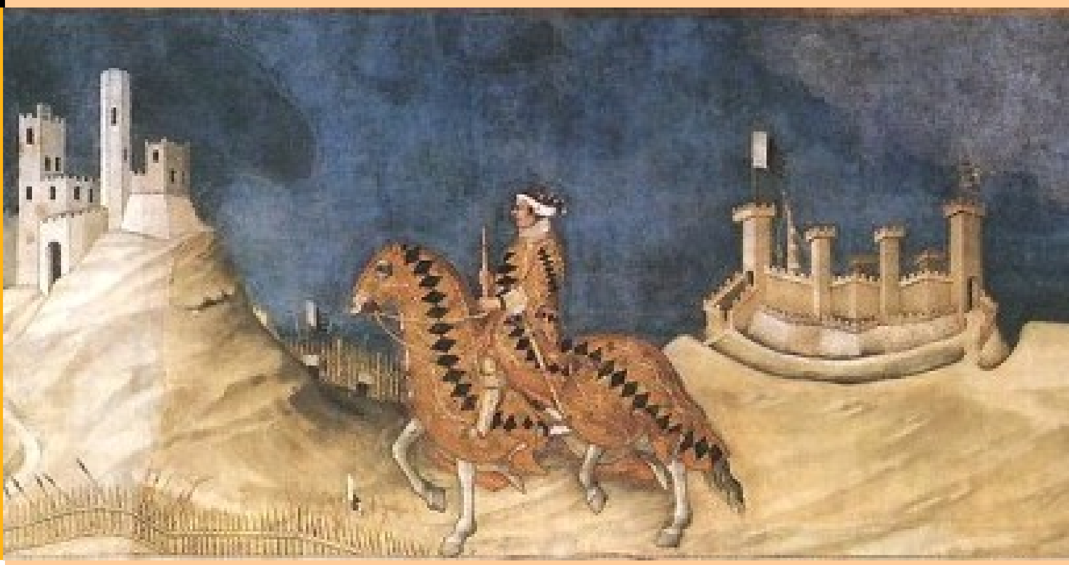
UNIVERSITÀ DI SIENA 1240

QUADERNI DEL DIPARTIMENTO
DI ECONOMIA POLITICA E STATISTICA

Stefano Vannucci

Symmetric Consequence Relations and Strategy-Proof Judgment Aggregation

n.754 – Maggio 2017



SYMMETRIC CONSEQUENCE RELATIONS AND STRATEGY-PROOF JUDGMENT AGGREGATION

STEFANO VANNUCCI

Abstract. It is shown that the posets of both substructural and classical symmetric consequence relations ordered by set-inclusion are (non-boolean) completely distributive complete lattices.

Therefore, those two basic versions of symmetric consequence relations are amenable to anonymous neutral and idempotent strategy-proof aggregation by majority polynomial rules on single-peaked domains. In particular, the majority rule is characterized as the only aggregation rule for odd profiles of symmetric consequence relations that is anonymous, bi-idempotent and strategy-proof on arbitrary rich locally unimodal domains.

MSC 2010 Classification: 05C05, 52021, 52037

JEL Classification Number: D71

1. Introduction

In recent years issues concerning judgment aggregation have attracted a wide interest among scholars from several disciplines including mathematics, logic, economics, theoretical computer science and artificial intelligence (see e.g. Konieczny and Pino-Perez (2002), Gärdenfors (2006), Dietrich (2007), Dietrich and List (2007), Everaere, Konieczny and Marquis (2007), Daniels and Pacuit (2008), Dokow and Holzman (2010), Grossi and Pigozzi (2014), Endriss (2016)).

Judgments are typically conceived of as proposition-like entities that may be interconnected and entertain relationships of mutual consistency or inconsistency. The current literature on judgment aggregation focusses on aggregation of sets of judgments, hence with theories of sorts.

In that connection, theories amount to sets of mutually consistent judgments that are closed with respect to certain *consequence relations* as defined on appropriate subsets of judgments.

Thus, at least part of the literature on judgment aggregation concerns in fact aggregation of theories, and consequence relations model indeed admissible *patterns of inference*. Therefore, arguably, consequence relations provide the underlying logical structure of such judgment sets or theories. But consequence relations may vary across agents, so when rephrased as theory aggregation, judgment aggregation involves aggregation of consequence relations. The present paper addresses the aggregation problem for consequence relations of a quite general variety, namely *symmetric consequence relations* (see e.g. Dunn and Hardegree (2001), and Shoesmith and Smiley (1978)), focussing on *strategy-proofness* properties of the available aggregation rules.

To this aim, the order-theoretic structure of two major classes of symmetric consequence relations is studied: it is shown that both of them are *completely distributive complete lattices* with respect to set-inclusion.

It is then proved that, as a corollary of the foregoing result, and for an odd number of agents, the *majority rule is in fact the only strategy-proof anonymous and bi-idempotent aggregation rule* for symmetric consequence relations on any rich domain of single peaked preferences.

2. Notation, definitions and results

2.1. **Preliminaries: Distributive Lattices and their Betweenness Relations.** Let $\mathcal{X}$ =$(X,\leqslant)$ be a partially ordered set of alternative (i.e. $\leqslant$ is a reflexive, transitive and antisymmetric binary relation on X). We denote as $x||y$ any pair x,y of $\leqslant$-incomparable elements.

A partially ordered set $\mathcal{X}=(X,\leqslant)$ is a **lattice** if both the least-upper-bound or *join* $\vee$ and the greatest-lower-bound or *meet* $\wedge$ of any $x,y\in X$ -as induced by $\leqslant$- are well-defined binary operations on X, and a **complete lattice** if the l.u.b $\vee Y$ and the g.l.b. $\wedge Y$ exist for all $Y\subseteq X$. A lattice $\mathcal{X}=(X,\leqslant)$ is **bounded** if there exist $\bot,\top\in X$ such that $\bot\leqslant x\leqslant\top$ for all $x\in X$: by definition, any complete lattice is bounded with $\bot=\vee\varnothing$, and $\top=\wedge\varnothing$. A lattice $\mathcal{X}=(X,\leqslant)$ is **distributive** if for any $x,y,z\in X$, $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$ (or, equivalently,$x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)$) i.e. the two equivalent *distributive identities* hold. A lattice $\mathcal{X}=(X,\leqslant)$ satisfies the **join-infinite distributive law** if for any x and $S\subseteq X$, $x\wedge(\bigvee S)=\bigvee\{x\wedge s:s\in S\}$, and the **meet-infinite distributive law** if for any x and $S\subseteq X$, $x\vee(\bigwedge S)=\bigwedge\{x\vee s:s\in S\}$. Notice that if a lattice $\mathcal{X}=(X,\leqslant)$ is complete then both the join-infinite distributive law and the meet-infinite distributive law imply distributivity of $\mathcal{X}$. A **frame** (**co-frame**) is a complete lattice $\mathcal{X}=(X,\leqslant)$ that satisfies the join-infinite distributive law (the meet-infinite distributive law) (see Johnstone (1982) for a classic, thorough introduction to frames). A distributive lattice $\mathcal{X}=(X,\leqslant)$ is **completely distributive** if for arbitrary families of elements of X it satisfies the two complete distributivity laws (CD-I) $\bigwedge_{i\in I}\bigvee_{j\in J}x_{i,j}=\bigvee_{f\in J^I}\bigwedge_{i\in I}x_{i,f(i)}$ and (CD-II) $\bigvee_{i\in I}\bigwedge_{j\in J}x_{i,j}=\bigwedge_{f\in J^I}\bigvee_{i\in I}x_{i,j}$. However, for *complete* lattices, it can be shown that CD-I and CD-II *are equivalent* (see e.g. Balbes and Dwinger (1974), chpt. XII). Thus, a **completely distributive complete lattice** is a complete lattice $\mathcal{X}=(X,\leqslant)$ that satisfies CD-I (or, equivalently, CD-II).

An **order filter** of a partially ordered set $\mathcal{X}$ is a set $F\subseteq X$ such that $z\in F$ if and only if $z\in X$ and $y\leqslant z$ for some $y\in F$: it is said to be *non-trivial* if $F\neq\varnothing$ and *proper* if $F\neq X$. An order filter F of a lattice $\mathcal{X}=(X,\leqslant)$ is a **(**latticial**) filter** if $x\wedge y\in F$ for any $x,y\in F$, and a **prime filter** if $x\vee y\in F$ implies that $x\in F$ or $y\in F$. The set of all *non-trivial and proper* prime filters of $\mathcal{X}$ is denoted by $\mathcal{F}_P$. It should be recalled here the following important and well-known fact to be used below: there is a bijection between the elements of a bounded distributive lattice $\mathcal{X}$ and the sets of prime filters of $\mathcal{X}$ they belong to, namely the function $\phi : X \to 2^{\mathcal{F}_P}$ defined by the rule $\phi(x) = \{F \in \mathcal{F}_P : x \in F\}$ is both injective and surjective (see e.g. Davey, Priestley (1990), chpt. 10).[1]

A ternary **betweenness** relation

$B_{\mathcal{X}} = \{(x,z,y) \in X^3 : x \wedge y \leqslant z \leqslant x \vee y\}$ is defined on $\mathcal{X}$,

which in turn induces an *interval function* $I^{B_{\mathcal{X}}} : X^2 \to 2^X$ such that and for any $x, y \in X$,

$I^{B_{\mathcal{X}}}(x,y) = B_{\mathcal{X}}(x,.,y) = \{z \in X : x \wedge y \leqslant z \leqslant x \vee y\}$ is the *interval* induced by x and y. Therefore, *for any* $x,y,z \in X$, $z \in B_{\mathcal{X}}(x,.,y)$ *if and only if* $(x,z,y) \in B_{\mathcal{X}}$ (*also written* $B_{\mathcal{X}}(x,z,y)$). The pair $(X, I^{B_{\mathcal{X}}})$ is the **interval space** induced by $B_{\mathcal{X}}$.

Moreover, a ternary **median** operation $\mu : X^3 \to X$ is defined on $\mathcal{X} = (X, \leqslant)$ by the following rule: for any $x,y,z \in X$,

$$\mu(x,y,z) = (x \wedge y) \vee (y \wedge z) \vee (x \wedge z) = (x \vee y) \wedge (y \vee z) \wedge (x \vee z)$$

(the latter identity is of course a consequence of distributivity). It is easily checked that for any $x,y,z \in X$, $\mu(x,y,z) = z$ if and only if $B_{\mathcal{X}}(x,z,y)$.

A few remarkable basic properties of $B_{\mathcal{X}}$ are listed below, and easily checked:

Claim 0. (see e.g. Nehring and Puppe (2007), Savaglio and Vannucci (2014)) The latticial betweenness relation $B_{\mathcal{X}}$ of a distributive lattice $\mathcal{X} = (X, \leqslant)$ satisfies the following conditions:

(i) *Symmetry*: for all $x,y,z \in X$, if $B_{\mathcal{X}}(x,z,y)$ then $B_{\mathcal{X}}(y,z,x)$;

(ii) *Closure* (or *Reflexivity*): for all $x,y \in X$, $B_{\mathcal{X}}(x,x,y)$ and $B_{\mathcal{X}}(x,y,y)$;

(iii) *Idempotence*: for all $x,y \in X$, $B_{\mathcal{X}}(x,y,x)$ only if $y = x$;

(iv) *Convexity* (or *Transitivity*): for all $x,y,z,u,v \in X$, if $B_{\mathcal{X}}(x,u,y)$, $B_{\mathcal{X}}(x,v,y)$ and $B_{\mathcal{X}}(u,z,v)$ then $B_{\mathcal{X}}(x,z,y)$;

(v) *Antisymmetry*: for all $x,y,z \in X$, if $B_{\mathcal{X}}(x,y,z)$ and $B_{\mathcal{X}}(y,x,z)$ then $x = y$.;

(vi) *Median-Equivalence*: for all $x,y,z \in X$, $B_{\mathcal{X}}(x,y,z)$ if and only if $\mu(x,y,z) = y$.

[1] That bijection is in fact the basis of Priestley's representation theorem, establishing that any bounded distributive lattice $\mathcal{X}$ is isomorphic to the lattice of all superset-closed clopen sets of the ordered topological space $(\mathcal{F}_P, \tau, \subseteq)$ where τ is the smallest topology on $\mathcal{F}_P$ which includes the set-theoretic union of $\{\{F \in \mathcal{F}_P : x \in F\} : x \in X\}$ and $\{\{F \in \mathcal{F}_P : x \notin F\} : x \in X\}$ (see e.g. Davey, Priestley (1990), Theorem 10.18).

2.2. **Symmetric consequence relations and their theories: structure.** Let $Z = Z_{\mathcal{L}}$ be a set of statements of an unspecified formal language $\mathcal{L}$, $2^Z = \{Y : Y \subseteq Z\}$ its power set, and $\vdash: 2^Z \times 2^Z \to 2$ a **Symmetric Consequence Relation (SCR)** on 2^Z as characterized by the two following properties::

$\vdash (i)$ (**Overlap**): for all $A, B \in 2^Z$ if $A \cap B \neq \varnothing$ then $\vdash (A, B) = 1$;

$\vdash (ii)$ **(Global Cut)**: for all $A, B, Y \subseteq Z$

if $\vdash (A, B) = 0$ then $\vdash (A \cup Y_1, B \cup Y_2) = 0$ for some $Y_1, Y_2 \subseteq Y$ such that $Y_1 \cup Y_2 = Y$ and $Y_1 \cap Y_2 = \varnothing$.

A SCR is *classical* if it also satisfies

$\vdash (iii)$ (**Weakening**): for all $A, B, C, D \in 2^Z$

if $\vdash (A, B) = 1$ then $\vdash (A \cup C, B \cup D) = 1$.

Conversely, a SCR that does *not* satisfy Weakening will also be referred to as *substructural* (see Dunn and Hardegree (2001) for a thorough discussion of the merits of symmetric consequence relations, and Restall (2000) for a comprehensive presentation of substructural logics and of the motivations underlying rejection of the classical Weakening condition[2]).

A *judgment set* or **theory** $T(\vdash)$ can be attached to any symmetric consequence relation $\vdash$by the following rule:

$$T(\vdash) = \left\{ \begin{array}{c} x \in Z : \ \vdash (\varnothing, \{x\}) = 1 \text{ or there exists } Y \subseteq Z \\ \text{such that } \vdash (Y, \{x\}) = \vdash (\varnothing, \{y\}) = 1 \text{ for each } y \in Y \end{array} \right\}.$$

It follows that a profile of symmetric consequence relations also defines a profile of judgment sets or theories.

Remark 1. One main reason for focussing on SCRs is related to valuations and semantics. *A* ***valuation*** on Z is a function $v : Z \to 2$, and the SCR $\vdash_V: 2^Z \to 2^Z$ **induced by a set** V **of valuations** on Z is defined as follows: for any $A, B \in 2^Z$, $\vdash_V (A, B) = 1$ if and only if for each $v \in V :\ v(y) = 1$ for some $y \in B$ whenever $v(z) = 1$ for every $z \in A$. Conversely, a valuation $v : Z \to 2$ **respects** a SCR $\vdash$ on Z if for all $A, B \in 2^Z$ such that $\vdash (A, B) = 1$, $v(y) = 1$ for some $y \in B$ whenever $v(z) = 1$ for each $z \in A$, and $V(\vdash)$ denotes the class of all valuations on Z that respect $\vdash$. A class V of valuations on Z is **sound** with respect to SCR $\vdash$ on Z if $\vdash^{-1} (1) \subseteq \vdash_V^{-1} (1)$, and a SCR $\vdash$ on Z is **complete** with respect to a class V of valuations on Z if $\vdash_V^{-1} (1) \subseteq \vdash^{-1}$ (1). A class V of valuations on Z is said to be a **semantics** for SCR $\vdash$ on Z if V characterizes $\vdash$, namely if $\vdash^{-1} (1) = \vdash_V^{-1} (1)$. It has been shown that SCRs -as opposed to ACRs- have not only a semantics but

[2]The main example of substructural symmetric consequence relations as defined in the present framework comes from *relevant logics* : $A \vdash B$ does not allow to conclude $A \cup C \vdash B$ because C may be *irrelevant* to B.

a unique semantics (i.e. SCRs satisfy not just **Completeness** but also **Absoluteness** as defined in Dunn and Hardegree (2001)).

Remark 2. A second main reason for focussing on SCRs is that working on the full domain $2^Z \times 2^Z$ makes it easier to compare the behaviour of binary consequence relations and classical Tarskian consequence operators. Indeed, a (classical) Tarskian consequence operator on Z is a closure operator on Z *i.e. a function* $k : 2^Z \to 2^Z$ that satisfies, for any $A, B \subseteq Z$: k-(i) (Extension): $A \subseteq k(A)$; k-(ii) (Monotonicity): if $A \subseteq B$ then $k(A) \subseteq k(B)$; k-(iii) (Idempotency): $k(k(A)) \subseteq k(A)$. A SCR $\vdash_k$on Z is induced by a closure operator k on Z by the following rule: for any $A, B \in 2^Z$, $\vdash_k (A, B) = 1$ if and only if $B = k(A)$. Notice, however, that now the intended meaning of $\vdash_k (A, B) = 1$ is that **the members of A jointly imply each member of B**. It is easily checked that any such $\vdash_k$is characterized by the following properties: for any $A, B, C, D \in 2^Z$,

C-Singularity (CSI): for each $A \in 2^Z$ there exists $B \in 2^Z$ such that $\vdash_k (A, B) = 1$ and $\vdash_k (A, C) = 0$ for any $C \neq B$;

C-Extension (CEX): if $\vdash_k (A, B) = 1$ then $A \subseteq B$;

C-Monotonicity (CMON) : if $A \subseteq B$, $\vdash_k (A, C) = 1$ and $\vdash_k (B, D) = 1$ then $C \subseteq D$;

C-Idempotency (CID): if $\vdash_k (A, B) = 1$ then $\vdash_k (B, B) = 1$.

Moreover, it is easily checked that $\vdash_k$ satisfies Global Cut but fails to satisfy both Overlap and Weakening. To check that Global Cut holds, observe that for any $Y, A, B \subseteq Z$, if $\vdash_k (A, B) = 0$ then either $B \subset k(A)$ or $k(A) \subset B$. If $B \subset k(A)$ then take $Y_1 = Y$ and $Y_2 = \varnothing$: thus, $B \cup \varnothing = B \subset k(A) \subseteq k(A \cup Y)$ hence $\vdash_k (A \cup Y_1, B \cup Y_2) = 0$. If on the contrary $k(A) \subset B$, then take $Y_1 = \varnothing$, $Y_2 = Y$: then $k(A \cup Y_1) = k(A) \subset B \subseteq B \cup Y$ whence $\vdash_k (A \cup Y_1, B \cup Y_2) = 0$, and Global Cut is satisfied. To see that Overlap obviously fails consider e.g. $A = \{p, q_p, q\}$ and $B = \{q\}$: $A \cap B \neq \varnothing$, but $\vdash_k (A, B) = 0$. To see that in general Weakening also fails, define constant closure operator $k^1 : 2^Z \to 2^Z$ by the rule $k^1(A) = Z$ for any $A \in 2^Z$, and take any closure operator $k \neq k^1$ (e.g. the identity closure operator k^{id}): then, by construction and CSI, $\vdash_k$violates Weakening.

A **closure-induced consequence relation (CICR)** on Z is a function $\vdash: 2^Z \times 2^Z \to 2$ that satisfies Global Cut, CSI, CEX, CMON, CID. Notice that any CICR $\vdash$ on Z induces a closure operator $k_\vdash : 2^Z \to 2^Z$ by the following rule: for any $A \in 2^Z$, $k_\vdash(A) = B$ where B is the only subset of Z such $\vdash (A, B) = 1$. Moreover, if $\vdash$ and $\vdash'$are distinct CICRs on Z then $k_\vdash \neq k_{\vdash'}$.

We denote by $\mathcal{C}_Z^s$ the set of all SCRs on Z, by $\mathcal{C}_Z^{cs}$ the set of all classical SCRs on Z, by $\mathcal{C}_Z^{\mathrm{csc}}$ the set of all closure-induced CICRs on Z, and by $\mathcal{C}_Z$ the set of all functions $\varphi : 2^Z \times 2^Z \to 2$, and assume $|Z| \geq 2$ in order to avoid the need for trivial qualifications. $\mathcal{C}_Z$ is partially ordered in the obvious way by partial order $\leqslant$ defined as follows: for any $\vdash_1, \vdash_2 \in \mathcal{C}_Z$,

$\vdash_1 \leqslant \vdash_2$ if and only if $\vdash_1^{-1}(1) \subseteq \vdash_2^{-1}(1)$ i.e. for each $A, B \subseteq Z$, if $\vdash_1 (A, B) = 1$ then $\vdash_2 (A, B) = 1$.

For any $D \subseteq \mathcal{C}_Z$, $\bigvee D = \vdash^{\vee D} \in C_Z$ such that, for any $A, B \subseteq Z$, $\vdash^{\vee D} (A, B) = 1$ iff there exists $\vdash_i \in C_Z$ with $\vdash_i (A, B) = 1$, and $\bigwedge D = \vdash^{\wedge D} \in C_Z$ such that, for any $A, B \subseteq Z$, $\vdash^{\wedge D} (A, B) = 1$ iff $\vdash_i (A, B) = 1$ for each $\vdash_i \in C_Z$. Thus, $\bigvee D$ is the least upper bound of D with respect to $\leqslant$, $\bigwedge D$ is the greatest lower bound of D with respect to $\leqslant$. In particular, for any $\vdash_1, \vdash_2 \in C_Z$, $\bigvee \{\vdash_1, \vdash_2\}$ is also written $\vdash_1 \vee \vdash_{2,}$and $\bigwedge \{\vdash_1, \vdash_2\}$ is also written $\vdash_1 \wedge \vdash_2$.

Clearly, $(\mathcal{C}_Z, \leqslant)$ is a completely distributive complete boolean (i.e. complemented) lattice, hence in particular a boolean frame and coframe.

Indulging in a slight abuse of language, we shall also use $\leqslant$ to also denote $\leqslant_{|\mathcal{C}_Z^s}, \leqslant_{|\mathcal{C}_Z^{cs}}, \leqslant_{|\mathcal{C}_Z^{cicr}}$, namely the restriction of $\leqslant$ to $\mathcal{C}_Z^s$, $\mathcal{C}_Z^{cs}$, $\mathcal{C}_Z^{cicr}$ respectively, since no ambiguity is likely to arise from this usage.

To begin with, we should be immediately that CICRs are an *antichain* with respect to $\leqslant$, namely

Claim 1. *Poset* $(\mathcal{C}_Z^{cicr}, \leqslant)$ *is the discrete order on* $\mathcal{C}_Z^{cicr}$ *, namely* $\leqslant$ *-when restricted to* $\mathcal{C}_Z^{cicr}$*- reduces to identity.*

Proof. Observe that for any $\vdash \in \mathcal{C}_Z^{cicr}$, $\vdash_1^{-1}(1)$ is precisely the graph of closure operator $k_\vdash$, and $|\vdash_1^{-1}(1)| = 2^{|Z|}$, by construction. It follows that for any $\vdash, \vdash' \in \mathcal{C}_Z^{cicr}$ if $\vdash \neq \vdash'$then $\vdash_1^{-1}(1) \neq \vdash_1'^{-1}(1)$ since $k_\vdash \neq k_{\vdash'}$. But then, $|\vdash_1^{-1}(1)| = |\vdash_1'^{-1}(1)|$ entails that $\vdash \nleqslant \vdash'$and $\vdash' \nleqslant \vdash$. $\square$

The corresponding order-theoretic structure induced by $\leqslant$ on $\mathcal{C}_Z^s$ and $\mathcal{C}_Z^{cs}$is, however, much richer and more regular. Indeed, we have the following

Theorem 1. *The posets* $(\mathcal{C}_Z^s, \leqslant)$ *and* $(\mathcal{C}_Z^{cs}, \leqslant)$ *are completely distributive complete lattices.*

Proof. To begin with, consider $\vdash^1, \vdash^0 \in C_Z$
defined as follows:
$\vdash^1 (A, B) = 1$ for any $A, B \subseteq Z$, while
$\vdash^0 (A, B) = 1$ if and only if $A \cap B \neq \varnothing$.

It is immediately checked that both $\vdash^1$and $\vdash^0$satisfy Overlap, Global Cut and Weakening, by definition, hence
$\vdash^0 \leqslant \vdash \leqslant \vdash^1$for any $\vdash \in \mathcal{C}_Z^s$. Therefore, both $(\mathcal{C}_Z^s, \leqslant)$ and $(\mathcal{C}_Z^{cs}, \leqslant)$ are bounded posets sharing their top and bottom elements.

Moreover, let $D = \{\vdash_i : i \in I\} \subseteq \mathcal{C}_Z^s$: $\bigvee D = \vdash^{\vee D} \in C_Z$ is defined by the following rule: for any $A, B \subseteq Z$, $\vdash^{\vee D} (A, B) = 1$ iff there exists $\vdash_i \in C_Z$ with $\vdash_i (A, B) = 1$.

Similarly, $\bigwedge D = \vdash^{\wedge D} \in C_Z$ is defined as follows: for any $A, B \subseteq Z$, $\vdash^{\wedge D} (A, B) = 1$ iff $\vdash_i (A, B) = 1$ for each $\vdash_i \in C_Z$.

By construction, both $\vdash^{\vee D}$and $\vdash^{\wedge D}$satisfy Overlap. Moreover, suppose that $D \neq \varnothing$ and $\vdash^{\wedge D} (A, B) = 0$ for some $A, B \subseteq Z$.

Then, there exists $\vdash_i \in D$ such that $\vdash_i (A, B) = 0$.

Hence, for each $Y \subseteq Z$ there exist $Y_1, Y_2 \subseteq Y$ such that $Y_1 \cap Y_2 = \varnothing, Y_1 \cup Y_2 = Y$, and $\vdash_i (A \cup Y_1, B \cup Y_2) = 0$, since $\vdash_i \in \mathcal{C}_Z^s$. But then, $\vdash^{\wedge D} (A \cup Y_1, B \cup Y_2) = 0$ i.e.

$\vdash^{\wedge D}$also satisfies Global Cut. Since $\vdash^1$is a maximum of $\mathcal{C}_Z^s$ with respect to $\leqslant$, it follows that $(\mathcal{C}_Z^s \leqslant)$ is a complete lattice.

Also, if $D = \{\vdash_i : i \in I\} \subseteq \mathcal{C}_Z^{sc}$, then $\vdash^{\wedge D}$satisfies Weakening, by construction. Therefore, $(\mathcal{C}_Z^{sc} \leqslant)$ is also a complete lattice.

Now, let I, J be two index sets, and $\{\vdash_{i,j}\}_{i \in I,, j \in J} \subseteq \mathcal{C}_Z^s$ ($\mathcal{C}_Z^{cs}$ respectively).

Then, both $\bigwedge_{i \in I} \bigvee_{j \in J} \vdash_{i,j}$and $\bigvee_{f \in J^I} \bigwedge_{i \in I} \vdash_{i,f(i)}$ exist and belong to $\mathcal{C}_Z^s$ ($\mathcal{C}_Z^{cs}$, respectively), by completeness.

Therefore, by definition,

$$\bigvee_{f \in J^I} \bigwedge_{i \in I} \vdash_{i,f(i)} \leqslant \bigwedge_{i \in I} \bigvee_{j \in J} \vdash_{i,j}.$$

Moreover, $\bigwedge_{i \in I} \bigvee_{j \in J} \vdash_{i,j} \leqslant \bigvee_{f \in J^I} \bigwedge_{i \in I} \vdash_{i,f(i)}$: indeed, for any $A, B \subseteq Z$, suppose $\bigwedge_{i \in I} \bigvee_{j \in J} \vdash_{i,j} (A, B) = 1$. Then, for each $i \in I$

there exists $j = \varphi(i) \in J$ such that $\vdash_{i,j} (A, B) = 1$, i.e. there exists $\varphi \in J^I$ such that $\bigwedge_{i \in I} \vdash_{i,\varphi(i)} (A, B) = 1$, whence

$$\bigvee_{f\in J^I}\bigwedge_{i\in I}\vdash_{i,f(i)}(A,B)=1.$$ □

Remark 3. It should be noticed that neither $(\mathcal{C}^s_Z,\leqslant)$ nor $(\mathcal{C}^{cs}_Z,\leqslant)$ nor are boolean i.e. (ortho-)complemented.

To see that $(\mathcal{C}^s_Z,\leqslant)$ is not (ortho-)complemented consider $C,D\subseteq Z$ such that $C\cap D=\varnothing$ and $\vdash^{(C,D)}$, $\vdash^{\uparrow(C,D)}\in C_Z$ defined as follows: for any $A,B\subseteq Z$,

$$\vdash^{(C,D)}(A,B)=1 \text{ iff either } A\cap B\neq\varnothing \text{ or } (A,B)=(C,D).$$

Clearly, $\vdash^{(C,D)}$satisfies Overlap by definition. To check that $\vdash^{(C,D)}$also satisfies Global Cut, take $A,B\subseteq Z$ such that $\vdash^{(C,D)}(A,B)=0$: then, $A\cap B=\varnothing$ and $(A,B)\neq(C,D)$ hence either $A\neq C$ or $B\neq D$, or both. If $A\neq C$, posit $Y_1=Y\cap A$ and $Y_2=Y\smallsetminus A$: then, $A\cup Y_1=A\neq C$, and $(A\cup Y_1)\cap(B\cup Y_2)=A\cap(B\cup(Y\smallsetminus A))=\varnothing$. Hence, $\vdash^{(C,D)}((A\cup Y_1)\cap(B\cup Y_2))=0$, as required. Thus, $\vdash^{(C,D)}\in\mathcal{C}^{sa}_Z\subseteq\mathcal{C}^{sa}_Z$.

Suppose that $(\mathcal{C}^{sa}_Z,\leqslant)$ is (ortho-)complemented, i.e. there exists a (unique) antitonic function $^*:\mathcal{C}^{sa}_Z\to\mathcal{C}^{sa}_Z$ such that for any $\vdash\in\mathcal{C}^{sa}_Z$: (i) $(\vdash^*)^*=\vdash$; (ii) $\vdash^*\wedge\vdash=\vdash^0$; (iii) $\vdash^*\vee\vdash=\vdash^1$. Hence, in particular, for any $A,B\subseteq X$ such that $A\cap B=\varnothing$ and $|B|=1$, $\vdash^*(A,B)=1$ iff $\vdash(A,B)=0$ (by (ii) and (iii)). Next, take $\vdash^{(A,B)}$for some disjoint $A,B\subseteq Z$ such that $A\cup B\neq Z$. By definition, $(\vdash^{(A,B)})^*(A,B)=0$. Then, consider $Y:=Z\smallsetminus(A\cup B)$ and any $Y_1,Y_2\subseteq Y$ such that $Y_1\cap Y_2=\varnothing$, $Y_1\cup Y_2=Y$ and $(A\cup Y_1)\cap(B\cup Y_2)=\varnothing$. By construction, either $A\cup Y_1\neq A$, or $B\cup Y_2\neq B$ (or both), hence $\vdash^{(A,B)}(A\cup Y_1,B\cup Y_2)=0$ and therefore $(\vdash^{(A,B)})^*(A\cup Y_1,B\cup Y_2)=1$. It follows that $(\vdash^{(A,B)})^*\notin\mathcal{C}^s_Z$ since it fails to satisfy Global Cut.

Now, for any $\vdash\in\mathcal{C}_Z$ define $\vdash^c\in C_Z$ as follows: for each $A,B\subseteq Z$, $\vdash^c(A,B)=1$ iff $A\cap B\neq\varnothing$ or $\vdash(A,B)=0$. Notice that, by definition, $\vdash\vee\vdash^c=\vdash^1$and $\vdash\wedge\vdash^c=\vdash^0$for each $\vdash\in\mathcal{C}_Z$: thus, the restriction of $^c:C_Z\to C_Z$ to $\mathcal{C}^{cs}_Z$ is the only possible (ortho-)complementation for $(\mathcal{C}^{cs}_Z,\leqslant)$. Clearly, $\vdash^c$ also satisfies Overlap by definition, for any $\vdash\in\mathcal{C}_Z$. However, for any $C,D\subseteq Z$, consider $\vdash^{\uparrow(C,D)}\in\mathcal{C}_Z$ defined as follows : for each $A,B\subseteq Z$,

$$\vdash^{\uparrow(C,D)}(A,B)=1 \text{ iff either } A\cap B\neq\varnothing \text{ or } [\ C\subseteq A \text{ and } D\subseteq B\ ].$$

It is immediately checked that $\vdash^{\uparrow(C,D)}$satisfies Overlap and Weakening, by definition. To check Global Cut, consider $A,B\subseteq Z$ such that $\vdash^{\uparrow(C,D)}(A,B)=0$. Thus, $A\cap B=\varnothing$ and $(A,B)\notin\uparrow(C,D)$ i.e. either $C\nsubseteq A$ or $D\nsubseteq B$. If $C\nsubseteq A$ then, for any $Y\subseteq X$, posit $Y_1:=Y\cap A$, and $Y_2:=Y\smallsetminus A$. Thus, $(A\cup Y_1)\cap(B\cup Y_2)=A\cap(B\cup(Y\smallsetminus A))=\varnothing$

and $C \nsubseteq A = A \cup Y_1$ hence $(A \cup Y_1) \cap (B \cup Y_2) \notin \uparrow (C, D)$. Therefore, $\vdash^{\uparrow(C,D)} ((A \cup Y_1), (B \cup Y_2)) = 0$ as required. It follows that $\vdash^{\uparrow(C,D)} \in \mathcal{C}_Z^{cs}$.

Finally, consider $(\vdash^{\uparrow(C,D)})^c$ for some $C, D \subseteq Z$ such that $C \cap D = \varnothing$ and $\varnothing \neq C \cup D \neq Z$, and $A, B \subseteq Z$ such that $A \cap B = \varnothing$, $\varnothing \neq A \cup B \neq Z$, and $(A \cup C) \cap (B \cup D) = \varnothing$. Then, by definition,

$\vdash^{\uparrow(C,D)} (A, B) = 0$ and $\vdash^{\uparrow(C,D)} (A \cup C, B \cup D) = 1$, while $(\vdash^{\uparrow(C,D)})^c(A, B) = 1$ and $(\vdash^{\uparrow(C,D)})^c(A \cup C, B \cup D) = 0$. Thus, $(\vdash^{\uparrow(C,D)})^c$ fails to satisfy Weakening whence $(\vdash^{\uparrow(C,D)})^c \notin \mathcal{C}_Z^{cs}$. It follows that $(\mathcal{C}_Z^{cs}, \leqslant)$ is not (ortho-)complemented.

Remark 4. SCRs are to be contrasted with **Asymmetric Consequence Relations (ACRs)**, that are much more widely used, but are only defined on those ordered pairs $(A, \{y\}) \in 2^Z \times 2^Z$ whose second element is a unit set. Substructural ACRs satisfy suitably adapted counterparts of Overlap and Global Cut. Classical ACRs also satisfy a suitably adapted counterpart of Weakening. Thus, ACRs may be regarded as **restrictions of SCRs to subdomain** $2^Z \times \{\{z\} : z \in Z\}$, that require a convenient reformulation of the basic axioms. It can be shown that a counterpart of the previous Theorem holds for the corresponding posets of substructural and classical ACRs. The details however will be spelled out elsewhere.

3. Symmetric consequence and strategy-proof judgment aggregation

Let $N = \{1, ..., n\}$ denote a finite population of agents, and $\mathcal{X} = (X, \leqslant)$ the relevant distributive lattice (frame) of alternative SCRs (i.e. $\leqslant$ is a reflexive, transitive and antisymmetric binary relation on X^*). We denote as $x||y$ any pair x, y of $\leqslant$-incomparable outcomes, and assume $|N| \geq 3$ in order to avoid tedious qualifications, where $|\cdot|$ denotes the cardinality of a set. Each agent in N proposes a symmetric consequence relation in X, and has a (possibly revealed) preference relation on X

Now, consider the set T_X of all *topped* preorders on X (i.e. reflexive and transitive binary relations having a unique maximum in X). For any $\succcurlyeq \in T_X$, $top(\succcurlyeq)$ denotes the unique maximum of $\succcurlyeq$ (while $\succ$ and $\sim$ denote the asymmetric and symmetric components of $\succcurlyeq$, respectively, and - for any $x \in X$ - $UC(\succeq, x) := \{y \in X : y \succcurlyeq x\}$ denotes the upper contour of $\succcurlyeq$ at x). *Single peaked* (total) preorders are those topped (total) preorders that 'respect' -i.e. are 'consistent with'- the betweenness relation $B_{\mathcal{X}}$. We shall focus on a very general notion of single peaked , labeled here as *local unimodality* and made precise by the following definition

Definition 1. *A topped preorder* $\succcurlyeq \in T_X$ *-with top outcome* x^**-is* ***locally unimodal*** *with respect to* $B_{\mathcal{X}}$ *(or* ***locally*** $B_{\mathcal{X}}$***-unimodal****) if and only if, for all* $y, z \in X$, $z \in B_{\mathcal{X}}(x^*, ., y)$ *implies that* $z \succcurlyeq y$*; moreover, for any* $Y \subseteq X$, $\succcurlyeq$ *is* Y***-complete*** *if for each* $y, y' \in Y$ *either* $y \succcurlyeq y'$ *or* $y' \succcurlyeq y$ *(or both), and* ***total*** *if it is* X*-complete.*

Let $U_{\mathcal{X}} \subseteq T_X$ denote the *set of all locally unimodal total preorders* (with respect to $B_{\mathcal{X}}$), and $U_{\mathcal{X}}^N$ the corresponding set of all N*-profiles* of locally unimodal total preorders or *full locally unimodal domain.* We shall mostly focus on locally unimodal domains of preorders that *need not be total but satisfy a suitable richness condition*, as made precise by the following definition:

Definition 2. *A set* $D_{\mathcal{X}}$ *of locally unimodal preorders (with respect to* $B_{\mathcal{X}}$*) is* ***rich*** *if for all* $x, y \in X$ *there exists* $\succcurlyeq \in D_{\mathcal{X}}$ *such that* $top(\succcurlyeq) = x$ *and* $UC(\succeq, y) = B_{\mathcal{X}}(x, ., y)$*.*

It should be noticed here that for each $x, y \in X$ one such rich locally unimodal preorder $\succcurlyeq_{x,y}^* \in U_{\mathcal{X}}$ with three indifference classes is easily defined as follows: take $\{x\}$, $B(x, ., y) \smallsetminus \{x\}$, and any subset of $X \smallsetminus B_{\mathcal{X}}(x, ., y)$ to be the top, middle, and bottom indifference classes of $\succcurlyeq^*$, respectively.

An **aggregation rule** for (N, X) is a function $f : X^N \to X$. For any $x_N \in X^N$, $y \in X$ and prime filter $F \in \mathcal{F}_P$, denote $N_y(x_N) = \{i \in N : y \leqslant x_i\}$ and, similarly, $N_F(x_N) = \{i \in N : x_i \in F\}$. The following properties of an aggregation rule will play a crucial role in the ensuing analysis.

Definition 3. *An aggregation rule* $f : X^N \to X$ *is* $B_{\mathcal{X}}$***-monotonic*** *if and only if for all* $x_N = (x_j)_{j \in N} \in X^N$, $i \in N$ *and* $x_i' \in X$, $f(x_N) \in B_{\mathcal{X}}(x_i, ., f(x_i', x_{N \smallsetminus \{i\}})$*.*

Definition 4. *An aggregation rule* $f : X^N \to X$ *is* ***monotonically independent (MI)*** *if and only if for all* $x_N, y_N \in X^N$ *and all* $x \in X$*: if* $N_x(x_N) \subseteq N_x(y_N)$ *then* $x \leqslant f(x_N)$ *implies* $x \leqslant f(y_N)$*.*

Remark 5. Thanks to Priestley's representation theorem for bounded distributive lattices (see Davey and Priestley (1990)) if $\mathcal{X} = (X, \leqslant)$ is a distributive lattice then the MI property can also be reformulated in terms of prime filters as follows: $f : X^N \to X$ is **monotonically independent (MI)** if and only if for all $x_N, y_N \in X^N$ and all $F \in \mathcal{F}_P$: if $N_F(x_N) \subseteq N_F(y_N)$ then $f(x_N) \in F$ implies $f(y_N) \in F$.

A **generalized committee** in N is a set of coalitions $\mathcal{W} \subseteq \mathcal{P}(N)$ such that $T \in \mathcal{W}$ if and only if $T \subseteq N$ and $S \subseteq T$ for some $S \in \mathcal{C}$ (a *committee* in N being a *non-empty* generalized committee in N which does *not* include the *empty* coalition)[3].

A **generalized committee aggregation rule** is a function $f : X^N \to X$ such that, for some fixed generalized committee $\mathcal{W} \subseteq \mathcal{P}(N)$ and for all $x_N \in X^N$, $f(x_N) = f_{\mathcal{W}}(x_N) := \vee_{S\in\mathcal{W}}(\wedge_{i\in S} x_i)$.[4]

A prominent example of a generalized committee voting rule is of course the ***majority rule*** f^{maj} defined as follows: for all $x_N \in X^N$, $f^{maj}(x_N) = \vee_{S\in\mathcal{W}^{maj}}(\wedge_{i\in S} x_i)$ where $\mathcal{W}^{maj} = \left\{ S \subseteq N : |S| \geq \lfloor \frac{|N|+2}{2} \rfloor \right\}$.

Claim 2. *Let $\mathcal{X} = (X, \leqslant)$ be a distributive lattice, and $\mathcal{W} \subseteq \mathcal{P}(N)$ a generalized committee in N. Then, the generalized committee aggregation rule $f_{\mathcal{W}} : X^N \to X$ is monotonically independent.*

Proof. Let $x_N, y_N \in X^N$ and $F \in \mathcal{F}_P$ such that $N_F(x_N) \subseteq N_F(y_N)$ and $f_{\mathcal{W}}(x_N) = \vee_{S\in\mathcal{W}}(\wedge_{i\in S} x_i) \in F$. By primality of F and finiteness of N, there exists $S \in \mathcal{W}$ such that $x_i \in F$ for each $i \in S$. Hence, $N_F(x_N) \subseteq N_F(y_N)$ implies that $S \subseteq \{i \in N : y_i \in F\} := S' \in \mathcal{W}$, by construction. But then, $(\wedge_{i\in S'} y_i) \in F$, whence $f_{\mathcal{W}}(y_N) = \vee_{S\in\mathcal{W}}(\wedge_{i\in S} y_i) \in F$ as required. □

Remark 6. Notice that the previous Claim fails if $\mathcal{X} = (X, \leqslant)$ is not a distributive lattice. To see this, consider the behaviour of the majority rule on M_3 or the **diamond** for $N = \{1, 2, 3\}$. The diamond, a 5-element lattice -consisting of the top element 1, the bottom element 0, and three mutually incomparable join-irreducible elements s_1, s_2, s_3- is the smallest **nondistributive modular lattice** (recall that a lattice $\mathcal{X} = (X, \leqslant)$ is **modular** iff for each $x, y, z \in X$, if $z \leqslant x$ then $x \wedge (y \vee z) = (x \wedge y) \vee z$). Then, take profiles $x_N = (1, s_2, s_3)$ and $y_N = (s_1, 0, 0)$.Observe that

$\{i \in N : s_1 \leq x_i\} = \{i \in N : s_1 \leq x_i\} = \{1\}$. Hence, by definition, for any monotonically independent $f : X^3 \to X$, $s_1 \leqslant f(x_N)$ if and only if $s_1 \leqslant f(y_N)$. However,

$f^{maj}(x_N) = (1 \wedge s_2) \vee (1 \wedge s_3) \vee (s_2 \wedge s_3) = 1$ hence $s_1 \leq f^{maj}(x_N)$, whereas

$f^{maj}(y_N) = (s_1 \wedge 0) \vee (s_1 \wedge 0) \vee (0 \wedge 0) = 0$ hence $s_1 \nleqslant f^{maj}(x_N)$.

Therefore, the majority rule f^{maj} is in general not monotonically independent on a non-distributive lattice (only projections i.e. collegial

[3]Thus, a generalized committee is just an *order filter* of the partially ordered set $(\mathcal{P}(N), \subseteq)$ of coalitions of N, while a committee is a *non-trivial* and *proper order filter* of $(\mathcal{P}(N), \subseteq)$ namely an order filter other than $\varnothing$ or $\mathcal{P}(N)$.

[4]On generalized committee rules on lattices and their properties see e.g. Monjardet (1990), Nehring and Puppe (2007), and Savaglio and Vannucci (2014).

aggregation rules- induced by a family of winning coalitions with a unique minimal coalition- are monotonically independent among non-constant rules: see Monjardet (1990), Theorem 3.4).

Remark 7. A **median** on set X is a ternary operation $f : X^3 \to X$ such that for any $x, y, z, u, v \in X$ the following properties hold:
(med-i): $f(x, x, y) = x$; (med-ii): $f(x, y, z) = f(y, x, z) = f(y, z, x)$; (med-iii) $f(f(x, y, z), u, v) = f(x, f(y, u, v), f(z, u, v))$.

A **median algebra** is a pair (X, f) where X is a set and f is a median on X (see e.g. Bandelt and Hedlíková(1983)).

A (ternary) **majority** on set X is a ternary operation $f : X^3 \to X$ such that for any $x, y \in X$ the following property holds:
(maj): $f(x, x, y) = f(x, y, x) = f(y, x, x) = x$.

Clearly, a median is also a majority, but not conversely.

A **majority algebra** is a pair (X, f) where X is a set and f is a (ternary) majority on X.

Now, let $\mathcal{X} = (X, \leqslant)$ be a lattice, and $\mu : X^3 \to X$ the ternary operation defined as follows: for any $x, y, z \in X$
$\mu(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$.

It is immediately seen that μ is a **majority** on X as defined above. If $\mathcal{X} = (X, \leqslant)$ is a **distributive lattice** it can be checked that μ is also a **median.** If on the contrary $\mathcal{X} = (X, \leqslant)$ is **not** distributive, however, μ is **not** a median on X. To check the latter statement, let us assume again that $\mathcal{X} = (X, \leqslant)$ is the **diamond** M_3 as defined in the previous Remark. Computation of the terms of identity (med-iii) with reference to the sequence $s_1, s_2, 1, s_3, 0$ yields:
$\mu(\mu(s_1, s_2, 1), s_3, 0) = \mu(1, s_3, 0) = s_3$, while $\mu(s_1, \mu(s_2, s_3, 0), \mu(1, s_3, 0)) = \mu(s_1, 0, 0) = 0 \neq s_3$
hence (med-iii) fails, as claimed.

Since aggregation rules only mention outcomes (as opposed to preferences on outcomes) their strategy-proofness properties require of course an explicit specification of the relevant preference domains. The ensuing analysis is mainly focussed on rich domains of locally unimodal preorders as made precise by the following definition:

Definition 5. *Let $f : X^N \to X$ be an aggregation rule and $D_{\mathcal{X}} \subseteq U_{\mathcal{X}}$ be a rich domain of locally unimodal and $f[X^N]$-complete*[5] *preorders (with respect to $B_{\mathcal{X}}$). Then, f is (individually)* ***strategy-proof*** *on $D^N_{\mathcal{X}}$ if and only if, for all $x_N \in X^N$, $i \in N$ and $x' \in X$, and for all $\succcurlyeq_N = (\succcurlyeq_j)_{j \in N} \in D^N_{\mathcal{X}}$, not $f(x', x_{N \setminus \{i\}}) \succ_i f(top(\succcurlyeq_i), x_{N \setminus \{i\}})$.*

[5]Here $f[X^N]$ denotes of course the range of f.

Lemma 1. *Let $\mathcal{X} = (X, \leqslant)$ be a bounded distributive lattice, $f : X^N \to X$ an aggregation rule for (N, X), and $D_{\mathcal{X}}$ a rich domain of locally $B_{\mathcal{X}}$-unimodal $f[X^N]$-complete preorders (with respect to $B_{\mathcal{X}}$). Then, the following statements are equivalent:*
(i) f is strategy-proof on $D^N_{\mathcal{X}}$;
(ii) f is $B_{\mathcal{X}}$-monotonic;
(iii) f is monotonically independent.

Proof. See Vannucci (2016a), Lemma 1. □

As a corollary of Theorem 1 and the foregoing Lemma, we have the following characterization result for the majority rule as an aggregation rule for symmetric consequence relations both substructural and classical.

Proposition 1. *Let B^s and B^{cs} the betweenness relations induced by the ternary median operations μ of $(\mathcal{C}^s_X, \leqslant)$ and μ' of $(\mathcal{C}^{cs}_X, I_{\mu'})$, respectively. Then, whenever N is odd-sized the corresponding majority rules $f^\mu : \mathcal{C}^{sN}_X \to \mathcal{C}^s_X$ and $f^{\mu'} : \mathcal{C}^{sN}_X \to \mathcal{C}^s_X$ are the only anonymous and bi-idempotent aggregation rules that are strategy-proof on any rich locally B^s-unimodal domain D of total preorders on $\mathcal{C}^s_X$, and on any rich locally B^{cs}-unimodal domain D of total preorders on $\mathcal{C}^{cs}_X$, respectively.*

Proof. Clearly, f^{maj} is anonymous by definition, and strategy-proof on D^N by Lemma 8, and the observation that f^μ and $f^{\mu'}$ are monotonically independent as implied by Claim 6. Moreover, if $|N|$ is odd then f^{maj} is bi-idempotent, by definition.

Conversely, suppose that $f : X^N \to X$ is anonymous, bi-idempotent and strategy-proof on D^N. Since f is strategy-proof on D^N, it follows by Lemma 1 that f is monotonically independent. But it can also be easily shown that (a) if f is (monotonically) independent and bi-idempotent then it is also neutral, and (b) if f is monotonically independent and neutral then f is a generalized committee aggregation rule i.e. there exists an order filter $\mathcal{W}$ of $(\mathcal{P}(N), \subseteq)$ such that for all $x_N \in X^N$
$f(x_N) = \vee_{A \in \mathcal{W}} \wedge_{i \in A} x_i$.[6]

Finally, anonymity of f entails that there exists a positive integer $k \leq |N|$ such that $\mathcal{W} = \{A \subseteq N : |A| \geq k\}$, and bi-idempotence of f implies that $|N| - k = k - 1$, whence $n = |N| = 2k - 1 = 2(k-1) + 1$ and $k = \frac{n+1}{2}$. Therefore, $\mathcal{W} = \left\{A \subseteq N : |A| \geq \frac{n+1}{2}\right\} = \mathcal{W}^{maj}$, namely $f = f^{maj}$. □

[6]Broadly speaking, the proof of points (a) and (b) sketched above is an adaptation and extension of a similar proof provided by Monjardet (1990) for *finite* distributive lattices (details are available from the author upon request).

4. Concluding remarks

The main results of the present paper establish that

(i) symmetric consequence relations are completely distributive complete lattices and therefore

(ii) symmetric consequence relations -and the theories (or judgment sets) attached to them- are amenable to anonymous, (bi-)idempotent and strategy-proof aggregation through the majority rule on a very general class of single peaked domains.

Since symmetric consequence relations (SCRs) may be construed as a representation of 'reasons' for the acceptance of the corresponding judgment sets, (ii) may be regarded as a positive result, especially from the perspective of *deliberative democracy,* that has exercised so many scholars in the last two or three decades. Of course, one might also be interested in coalitional strategy-proofness properties of the majority rule on the same single peaked preference domains for SCRs. Unfortunately, however, the majority rule fails to satisfy coalitional strategy-proofness on the lattices of SCRsconsidered above, due to the incidence-geometric structure of the interval spaces induced by their betweenness relations (see Vannucci (2016)). Finally, as noticed above, there are plenty of SCRs that are arguably implausible or weird: it remains to be seen whether some 'reasonable' (distributive) sublattices of SCRs do exist and can be conveniently characterized.

References

[1] Balbes R., P. Dwinger (1974) : *Distributive Lattices.* University of Missouri Press, Missouri.

[2] Bandelt H.-J., J. Hedlíková (1983): Median algebras, *Discrete Mathematics* 45, 1-30.

[3] Daniels T., E. Pacuit (2008): A general approach to aggregation problems, *Journal of Logic and Computation* 19, 517-536.

[4] Davey B.A, H.A. Priestley (1990): *Introduction to Lattices and Order.* Cambridge University Press, Cambridge.

[5] Dietrich F. (2007): A generalized model of judgment aggregation, *Social Choice and Welfare* 28, 529-565.

[6] Dietrich F., C. List (2007): Strategy-proof judgment aggregation, *Economics and Philosophy* 23, 269-300.

[7] Dokow E., R. Holzman (2010): Aggregation of non-binary evaluations, *Advances in Applied Mathematics* 45, 487-504.

[8] Dunn J.M., G.M. Hardegree (2001): *Algebraic Methods in Philosophical Logic.* Oxford Logic Guides 41, Clarendon Press, Oxford.

[9] Endriss U. (2016): Judgment Aggregation, in F.Brandt, V.Conitzer, U.Endriss, J.Lang, A.D. Procaccia (eds.): *Handbook of Computational Social Choice.* Cambridge University Press, Cambridge.

[10] Everaere P., S. Konieczny, P. Marquis (2007): The strategy-proofness landscape of merging, *Journal of Artificial Intelligence Research* 28, 49-105.

[11] Gärdenfors P. (2006): An Arrow-like theorem for voting with logical consequences, *Economics and Philosophy* 22, 181–190.

[12] Grossi D., G. Pigozzi (2014): *Judgment Aggregation: A Primer.* Synthesis Lectures on Artificial Intelligence and Machine Learning. Morgan and Claypool Publishers, Milton Keynes UK.

[13] Johnstone P. (1982): *Stone Spaces.* Cambridge University Press, Cambridge.

[14] Konieczny S., R. Pino-Perez (2002): Merging information under constraints: a logical framework, *Journal of Logic and Computation* 12, 773-808.

[15] Monjardet B. (1990): Arrowian characterizations of latticial federation consensus functions, *Mathematical Social Sciences* 20, 51-71.

[16] Nehring K., C. Puppe (2007): The structure of strategy-proof social choice - Part I: General characterization and possibility results on median spaces, *Journal of Economic Theory* 135, 269-305.

[17] Restall G. (2000): *An Introduction to Substructural Logics.* Routledge, London.

[18] Savaglio E., S. Vannucci (2014): Strategy-proofness and single peakedness in bounded distributive lattices, arXiv:1406.512v1 [q-fin.EC].

[19] Shoesmith D.J., T.J. Smiley (1978): *Multiple-Conclusion Logic.* Cambridge University Press, Cambridge.

[20] Vannucci S. (2016): Weakly unimodal domains, anti-exchange properties and coalitional strategy-proofness of aggregation rules, *Mathematical Social Sciences* 84, 56-67.

[21] Vannucci S. (2016a): Majority judgment and strategy-proofness, Quad. DEPS 730, Università di Siena.

Department of Economics and Statistics, University of Siena